Journal of Statistical Physics, Vol. 98, Nos. 5/6, 2000

Kinetic Theory of Area-Preserving Maps. Application to the Standard Map in the Diffusive Regime

R. Balescu[1]

Received July 12, 1999; final September 3, 1999

The evolution of the distribution function of a dynamical system governed by a general two-dimensional area-preserving iterative map is studied by the methods of nonequilibrium statistical mechanics. A closed, non-Markovian master equation determines the angle-averaged distribution function (the "density profile"). The complementary, angle-dependent part ("the fluctuations") is expressed as a non-Markovian functional of the density profile. Whenever there exist two widely separated intrinsic time scales, the master equation can be markovianized, yielding an asymptotic kinetic equation. The general theory is applied to the standard map in the diffusive regime, i.e., for large stochasticity parameter and large scale length. The non-Markovian master equation can be written and solved analytically in this approximation. The two characteristic time scales are exhibited. This permits the thorough study of the evolution of the density profile, its tendency toward the Markovian approximation, and eventually toward a diffusive Gaussian packet. The evolution of the fluctuations is also described in detail. The various relaxation processes are governed asymptotically by a single diffusion coefficient, which is calculated analytically. This model appears as a testing bench for the study of kinetic equations. The various previous approaches to this problem are reviewed and critically discussed.

KEY WORDS: Iterative maps; standard map; nonequilibrium statistical mechanics; kinetic equations; non-Markovian effects; non-Gaussian effects; diffusive regime; diffusion coefficient.

1. INTRODUCTION

Iterative maps have been extensively used for the study of evolution problems, possibly as a substitute of differential equations. They lead to a

---

[1] Association Euratom-Etat Belge pour la fusion, Université Libre de Bruxelles, CP 231 Campus Plaine ULB, 1050 Brussels, Belgium; e-mail: rbalescu@ulb.ac.be.

much simpler formalism, which is particularly useful in numerical calculations. They may either be derived from differential equations by the Poincaré section or stroboscopic section techniques, or else they may be constructed a priori. Of particular importance for the modelling of classical mechanical systems are *Hamiltonian* (*or area-preserving*) *maps*. They preserve the main mathematical structure of continuous-time dynamics. In particular, the values of the dynamical variables at time $\tau+1$ are related to those at time τ by a canonical transformation. We limit ourselves in the present work to *two-dimensional area-preserving maps*, depending on a single, non-negative parameter K, the *stochasticity parameter*. We assume that one of the variables is an *angle* denoted, for convenience, by $2\pi\theta$; the other co-ordinate (analogous to an *action* variable) is denoted by x. A general map of this type is defined by the following recurrence relations:

$$x_{\tau+1}=x_\tau+\Xi(x_\tau, \theta_\tau; K), \qquad \theta_{\tau+1}=\theta_\tau+\Theta(x_\tau, \theta_\tau; K) \tag{1}$$

Here τ is the *discrete time*, taking integer values 0, 1, 2, ...; x_τ, θ_τ are, respectively, the values of the variables x, θ at time τ. Ξ and Θ are continuous functions of their three variables x, θ, K. They are periodic in θ, with period 1; this variable may thus be restricted to the range (0–1). These functions satisfy some constraints ensuring the Hamiltonian character of the map; we do not write down here these constraints.$^{(1)}$

In the first part of this paper we consider properties of maps of the general form (1). These properties will be illustrated for the specific case of the well-known Chirikov–Taylor *standard map*$^{(2-4)}$ (we only quote here a few among the numerous works devoted to this subject). It is, indeed, one of the simplest two-dimensional Hamiltonian maps of the class (1), many properties of which can be derived analytically:

$$\begin{aligned} x_{\tau+1} &= x_\tau - \frac{K}{2\pi}\sin 2\pi\theta_\tau \\ \theta_{\tau+1} &= \theta_\tau + x_{\tau+1}, \qquad (\mathrm{mod}\ 1) \end{aligned} \tag{2}$$

Iterative (in particular, Hamiltonian) maps prove to be useful tools for the study of transport processes. In order to treat such problems one adopts a *statistical description* (see, e.g., refs. 5 or 6, Chap. 15). The consideration of individual trajectories defined by Eqs. (2) is then replaced by the study of a statistical ensemble defined by a distribution function $F(x, \theta; \tau)$ in the phase space spanned by the variables x and θ: this is a 1-periodic function of θ and is defined only for integer values of τ. [The distribution function also depends on the parameter K; for brevity, this dependence will most often not be written down explicitly.] The relation between the individual orbits point of view described by Eqs. (1) or (2) and the statistical point of view is analogous to the relation between trajectories defined in continuous time by Hamilton's equations of motion and the description in terms of a distribution function obeying the Liouville equation.

Of special physical interest for transport theory is the angle-averaged distribution function, called the *density profile* $n(x;\tau)$:

$$n(x;\tau)=\int_0^1 d\theta\, F(x,\theta;\tau) \tag{3}$$

It has been known for a long time$^{(2-4,\,7-14)}$ that in the limit of large K, the evolution described by the standard map has a *diffusive* character. This statement has to be made more precise, because it may address various aspects of the evolution. In most previous works the purpose was to show (analytically or numerically) that the second moment (mean square displacement) of the density profile grows asymptotically linearly in time, and to calculate the corresponding diffusion coefficient. These works will be reviewed and discussed in Section 11. A complete proof of the existence of a diffusive regime involves the study of the complete density profile. It should show that the density profile, starting from an arbitrary initial state and evolving by the standard map dynamics, tends asymptotically toward a Gaussian packet, which is the solution of a diffusion equation. This is the goal of our present study.

In "classical" statistical mechanics such a study involves the solution of a *Master equation*, i.e., a *closed* equation for a reduced distribution function. A corresponding equation for systems described by discrete-time iterative maps was obtained in a recent paper by Bandtlow and Coveney.$^{(15)}$ They derived an *exact* closed equation for the density profile, analogous to the Master equation obtained by Prigogine and Résibois$^{(16)}$ and by Zwanzig$^{(17)}$ in continuous-time statistical mechanics. The most important characteristic of both equations is their *non-Markovian nature*: the evolution of the system at time τ is determined not only by its instantaneous state, but rather by its past history. It is well known in continuous-time kinetic theory that, whenever there exist two characteristic time scales that are widely separated (e.g., the duration of a collision, and the inverse collision frequency in a gas) the Master equation reduces, for times much longer than the short time scale, to a *Markovian kinetic equation.*

The Bandtlow–Coveney equation is quite general; it appears that the standard map provides us with an ideal testing bench for studying its properties. It is interesting to investigate whether there exist here also two such characteristic time scales, and under which conditions a markovianization is justified. This will be one of the objects of the present work.

In order to obtain a complete description of the system, an equation for the complementary, angle-dependent part of the distribution function is also needed. More precisely, the distribution function is decomposed as follows:

$$F(x, \theta; \tau) = n(x; \tau) + G(x, \theta; \tau) \tag{4}$$

where the function G has vanishing angle-average:

$$\int_0^1 d\theta \, G(x, \theta; \tau) = 0 \tag{5}$$

In continuous-time statistical mechanics the role of G is played by the correlations. The formalism of non-equilibrium statistical mechanics results in a complete description of the correlations.(18) Here too, the basic equation for G is non-Markovian. But under the same conditions as described above, a Markovian regime is reached asymptotically, in which the correlations at time τ become functionals of the average distribution function at the same time. In the present work this formalism will be extended to discrete-time dynamical systems.

The paper is organized as follows. After a brief review of continuous-time statistical mechanics (Section 2), the formalism is extended to discrete-time systems in Section 3, following and completing the work of ref. 15. The Master equation for the density profile and the equation for the fluctuations are established for a general 2-dimensional area-preserving map. These equations are applied specifically to the standard map for arbitrary values of the stochasticity parameter in Section 4. They take a simpler, analytically tractable form in the "diffusive regime", as shown in Sections 5 and 6. The non-Markovian Master equation is solved for the density profile and its relation to the Markovian approximation is studied in Section 7. Even the Markovian solution remains non-Gaussian for a long time before reaching the truly diffusive Gaussian form (Section 8). The role of the initial fluctuations in the evolution of the density profile is studied in Section 9. Finally Section 10 is devoted to the evolution of the fluctuations. Section 11 contains a review and a critical discussion of previous works, ending with the conclusions (Section 12).

2. REVIEW OF CONTINUOUS-TIME NON-EQUILIBRIUM STATISTICAL MECHANICS

We consider a classical dynamical system of N particles, inscribed in a phase space whose coordinates are the positions q_j and the momenta p_j

of the particles, and characterized by a Hamiltonian $H(\mathbf{X})$. We introduce here the abbreviation $\mathbf{X} = (q_1, p_1, ..., q_N, p_N)$. The state of the system is described statistically by a distribution function $F(\mathbf{X}; t)$, obeying the Liouville equation:

$$\frac{\partial}{\partial t} F(\mathbf{X}; t) = \mathscr{L} F(\mathbf{X}; t) \tag{6}$$

where the Liouvillian operator is the Poisson bracket with the Hamiltonian: $\mathscr{L} F = [H, F]$. The solution of the initial-value problem of this equation can be written formally in terms of a propagator:

$$F(\mathbf{X}; t) = \mathscr{G}(t)\, F(\mathbf{X}; 0) \equiv \exp(\mathscr{L} t)\, F(\mathbf{X}; 0) \tag{7}$$

We now consider a decomposition of the distribution function into two *mutually exclusive* parts, called the *vacuum* and the *correlations*. The explicit representation of this decomposition is not important here. Various definitions can be adopted: one of them is based on a Fourier decomposition(19, 20); alternatively, a representation in terms of reduced distribution functions is closer to the physical concept of correlations.(6, 18) The mutual exclusion of the two components allows us to introduce two projection operators V and C, with the following properties: $V + C = I$, $V^2 = V$, $C^2 = C$, $VC = CV = 0$. The decomposition of the distribution function is then written as:

$$F(\mathbf{X}; t) = VF(\mathbf{X}; t) + CF(\mathbf{X}; t) \tag{8}$$

Some straightforward calculations which we do not reproduce here, because they are very similar to those described in the next section (see ref. 20, Section 16.3) lead, without any approximation, to the following conclusion. The vacuum component of the distribution function, $VF(t)$, is shown to obey the following equation (omitting to write the variable $\mathbf{X}$):

$$\frac{\partial}{\partial t} VF(t) = C\mathscr{L} V(t) + \int_0^t ds\, V\mathscr{E}(s)\, VF(t-s) + V\mathscr{D}(t)\, CF(0) \tag{9}$$

where $\mathscr{E}(t)$, $\mathscr{D}(t)$ are operators defined in terms of the Liouvillian and called, respectively, the irreducible evolution operator and the destruction operator (again, we need not specify their form here). This is the well-known *Master Equation* due to Prigogine and Résibois. (16, 17) It is a *closed equation for the vacuum component of the distribution function.* Its main characteristics are the following:

- It is a *non-Markovian Equation*: through the convolution appearing in the second term of the right hand side, the rate of change of the unknown function at time t is related to the whole past history, back to $t=0$.
- The third term in the right hand side is related to the *initial value of the complementary part* $CF(0)$: this appears as a source term in the Master equation.
- The first term in the equation is a *Markovian* contribution determined by the action of the diagonal vacuum part of the Liouvillian operator; for a gas of interacting particles this term corresponds to the mean field, or *Vlasov* operator.

The first two features are not compatible, at first sight, with the known macroscopic laws of continuum mechanics. One of the important purposes of non-equilibrium statistical mechanics has been to understand how this compatibility can be restored. Whenever some conditions are satisfied, it appears that an asymptotic regime sets in, in which the evolution is described by simpler laws. This occurs, in particular, whenever there exist two widely separated intrinsic time scales in the system. In a dilute gas, for instance, these are the duration of a collision and the much longer mean free time between collisions. The first one is related to the rate of change of the operators $V\mathscr{E}(t)\,V$, $V\mathscr{D}(t)\,C$: it is a measure of the effective range of the memory. The second one characterizes the rate of change of the distribution function $VF(t)$: it describes the rate of relaxation towards the final (equilibrium) state. Given that the distribution function varies very slowly on the short time scale, the following asymptotic limit, called *Markovianization* can be performed on the Master equation:

- The destruction term, which tends to zero for times longer than the short time scale, is neglected;
- In the second term the retardation of the distribution function is neglected: $VF(t-s)\to VF(t)$, and the integration is extended to infinity (because the integrand is cut off by $V\mathscr{E}(s)\,V$).

We are then left with the following *Markovian* equation, called the *Kinetic Equation*:

$$\frac{\partial}{\partial t}\,VF(t)=V\mathscr{L}V(t)+V\mathbb{E}VF(t) \tag{10}$$

with the following time-independent "collision operator":

$$V\mathbb{E}V=\int_0^\infty ds\;V\mathscr{E}(s)\;V \tag{11}$$

For a dilute gas, Eq. (10) is the well-known Boltzmann kinetic equation. In order to complete the picture, the correlation part $CF(t)$ is treated in a similar way (see Section 3); as a result, the following equation is obtained:

$$CF(t) = \int_0^t ds\, C\mathscr{C}(s)\, VF(t-s) + C\mathscr{P}(t)\, CF(0) \tag{12}$$

where $C\mathscr{C}(t)\, V$, $C\mathscr{P}(t)\, C$ are called, respectively, creation fragment and propagation fragment. This equation shows that the correlation component of the distribution function is a functional of the vacuum. The relation is, however, also "non-Markovian" in the sense that the correlation at time t is determined by the whole past history of the vacuum part. It also contains a contribution of the initial correlation, propagated by the last term in the right hand side.

Whenever the two time scales defined above are widely separated, this exact relation can also be Markovianized:

$$CF(t) = C\mathbb{C}VF(t) \tag{13}$$

with:

$$C\mathbb{C}V = \int_0^\infty ds\, C\mathscr{C}(s)\, V \tag{14}$$

The validation of the assumptions leading from the exact Master equations (9) and (12) to the asymptotic kinetic equations (10) and (13) is not always an easy matter for realistic systems. It is therefore interesting to note that for some simple model systems, such as the standard map, this passage can be studied in considerable detail, at least in a certain limit. This will be shown in the forthcoming sections. Before doing that, the formalism of non-equilibrium statistical mechanics has to be adapted to the dynamical systems described in discrete time by an area-preserving map.

3. DISCRETE-TIME NONEQUILIBRIUM STATISTICAL MECHANICS

We consider a dynamical system whose evolution is defined by an area-preserving map of the form (1); its statistical state is determined by a distribution function $F(x, \theta; \tau)$ defined in Section 1. The evolution of this function can no longer be determined by a differential equation, such as the

Liouville equation (6), because the time derivative has no meaning in discrete time. A relation like (7) remains, however, relevant. One therefore associates with the map (1) the *Perron–Frobenius operator*, $\mathcal{U}$, defined by the following recurrence relation:

$$F(x, \theta; \tau+1) = \mathcal{U} F(x, \theta; \tau) \tag{15}$$

Alternatively, one may introduce the *propagator*, which relates the instantaneous distribution function to its initial value:

$$F(x, \theta; \tau) = \mathcal{U}^{\tau} F(x, \theta; 0) \tag{16}$$

In continuous-time dynamics the Perron–Frobenius operator $\mathcal{U}$ corresponds to $\exp(\mathcal{L})$, and the propagator $\mathcal{U}^{\tau}$ corresponds to the propagator $\exp(\mathcal{L}\tau)$ defined in Eq. (7). (Here and below, roman letters $t, s,\ldots$ denote real, continuous time variables, whereas greek letters $\tau, \sigma,\ldots$ denote discrete time variables, taking only integer values.) The properties of the Perron–Frobenius operator have been described in many places.(5, 6, 21, 22)

The Fourier transform of the distribution function with respect to both phase space variables, $f_M(q; \tau)$, will be extensively used below:

$$F(x, \theta; \tau) = \sum_{M=-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, e^{2\pi i(qx+M\theta)} f_M(q; \tau) \tag{17}$$

The statistical state of the system is thus completely specified by the set of all Fourier components. The latter "*distribution vector*" will be represented by the single symbol $f(\tau)$, denoting the set of functions $\{f_M(q, \tau),\ M=0, \pm 1, \pm 2,\ldots\}$. From here on, the following convention will be used. We denote by a capital letter: $M, N,\ldots$ an arbitrary *integer*, positive, negative, *or zero*; we denote by a lower-case letter: $m, n,\ldots$ an arbitrary *non-zero integer*.

As explained in Section 1, we are interested in deriving an equation of evolution for the angle-averaged distribution function, or *density profile* $n(x; \tau)$ [Eq. (3)] or, equivalently, for its Fourier transform, which is simply the $M=0$ Fourier component of the distribution function; it will be denoted by the notation $\varphi(q; \tau)$:

$$\begin{aligned} n(x; \tau) &= \int_{-\infty}^{\infty} dq\, e^{2\pi i q x} \varphi(q; \tau) \\ \varphi(q; \tau) &= f_0(q; \tau) \end{aligned} \tag{18}$$

The following important properties are immediately derived from the definition (17). The normalization of the distribution function $F(x, \theta; \tau)$ is expressed as follows in the Fourier representation:

$$\int_{-\infty}^{\infty} dx \int_0^1 d\theta\, F(x, \theta; \tau) = \varphi(0, \tau) = 1 \tag{19}$$

The moments of the density profile $n(x; \tau)$ are simply expressed in terms of the derivatives of its Fourier transform:

$$\langle x^p(\tau) \rangle = \frac{1}{(2\pi i)^p} \frac{\partial^p \varphi(q; \tau)}{\partial q^p}\bigg|_{q=0} \tag{20}$$

The (Fourier) density profile can also be obtained by acting on the full distribution function with a projection operator P whose effect is the average over the angle θ [or the extraction of the $M=0$ component of the Fourier series (17)]:

$$Pf(\tau) = f_0(q; \tau) \equiv \varphi(q; \tau) \tag{21}$$

(In forthcoming equations, the arguments q, x, θ of the distribution functions will not be written down explicitly whenever they are clearly understood.) Obviously, $P^2 = P$. Let Q be the complement of the projector P, thus: $P + Q = I$, (where I is the identity operator), and $PQ = QP = 0$. The operators P and Q thus form a pair of projectors, analogous to the operators V and C of Section 2. By the action of these projectors, one produces a decomposition of the distribution function analogous to Eq. (8):

$$f(\tau) = Pf(\tau) + Qf(\tau) \tag{22}$$

The *fluctuation part* $Qf(\tau)$ comprises the set of all Fourier components with $m \neq 0$; in order to stress this constraint, we choose a different notation for the Fourier components with $m \neq 0$:

$$\begin{aligned} g_m(q; \tau) &\equiv f_m(q; \tau), \qquad m \neq 0 \\ g_0(q; \tau) &= 0 \end{aligned} \tag{23}$$

Thus, $Qf(\tau)$, also denoted by the simpler notation $g(\tau)$, represents the following *set* of functions:

$$Qf(\tau) \equiv g(\tau) = \{g_m(q; \tau); m = \pm 1, \pm 2\} \tag{24}$$

We now wish to derive a closed equation of evolution for the density profile, and a separate equation for the set of fluctuations. Although many of these calculations can be found in refs. 6, 14, 15, 21, we briefly sketch them here for the reader's convenience. Bandtlow and Coveney$^{(15)}$ start from the trivial identity expressing the group property of the Perron–Frobenius operator:

$$\mathscr{U}^{\tau+1} = \mathscr{U}\mathscr{U}^{\tau} \tag{25}$$

which is projected on the P and Q subspaces, and rewritten in the form:

$$\begin{aligned} P\mathscr{U}^{\tau+1} &= P\mathscr{U}P\mathscr{U}^{\tau} + P\mathscr{U}Q\mathscr{U}^{\tau} \\ Q\mathscr{U}^{\tau+1} &= Q\mathscr{U}P\mathscr{U}^{\tau} + Q\mathscr{U}Q\mathscr{U}^{\tau} \end{aligned} \tag{26}$$

We now perform a so-called Z-*transformation*,$^{(15)}$ which plays the same role for discrete-time systems as the Laplace transformation in continuous time. This is simply a Laurent series with non-positive exponents. The Z-transform of an arbitrary function (or operator) $\mathscr{T}(\tau)$ is defined as:

$$\mathscr{Z}[\mathscr{T}(\tau)] \equiv \tilde{\mathscr{T}}(z) = \sum_{\tau=0}^{\infty} \mathscr{T}(\tau)\, z^{-\tau} \tag{27}$$

where z is a complex variable. In particular, when $\mathscr{T}(\tau) = \mathscr{T}^{\tau}$, the geometrical series is summed in the form:

$$\mathscr{Z}(\mathscr{T}^{\tau}) = \frac{z}{z-\mathscr{T}} \tag{28}$$

We now note the following important properties of the Z-transform, which are easily proved. The *inverse Z-transformation* is:

$$\mathscr{Z}^{-1}[\tilde{\mathscr{T}}(z)] \equiv \mathscr{T}(\tau) = \frac{1}{2\pi i} \oint_C dz\, z^{-\tau-1} \tilde{\mathscr{T}}(z) \tag{29}$$

where C is a closed contour surrounding all singularities of $\tilde{\mathscr{T}}(z)$.[2] The following relation is called the *shifting theorem*:

$$\mathscr{Z}(\mathscr{T}(\tau+\sigma)) = z^{\sigma} \left[\tilde{\mathscr{T}}(z) - \sum_{\tau=0}^{\sigma-1} \mathscr{T}(\tau)\, z^{-\tau} \right] \tag{30}$$

---

[2] Hasegawa and Saphir$^{(6, 14, 21)}$ used Eq. (29) without referring explicitly to the Z-transformation.

We finally note the *convolution theorem* in discrete time:

$$\mathscr{Z}\left\{\sum_{\sigma=0}^{\tau} \mathscr{S}(\sigma)\, \mathscr{T}(\tau-\sigma)\right\} = \tilde{\mathscr{S}}(z)\, \tilde{\mathscr{T}}(z) \tag{31}$$

We now return to Eq. (26) both sides of which are multiplied by $z^{-\tau}$ and summed over τ from 0 to ∞. In the left hand side we make use of the shifting theorem (30) and obtain two relations for the Z-transformed propagator:

$$\begin{aligned} z[P\tilde{\mathscr{U}}(z) - P\mathscr{U}^0] &= P\mathscr{U}P\tilde{\mathscr{U}}(z) + P\mathscr{U}Q\tilde{\mathscr{U}}(z) \\ z[Q\tilde{\mathscr{U}}(z) - Q\mathscr{U}^0] &= Q\mathscr{U}P\tilde{\mathscr{U}}(z) + Q\mathscr{U}Q\tilde{\mathscr{U}}(z) \end{aligned} \tag{32}$$

The second equation is solved for $Q\tilde{\mathscr{U}}(z)$, and the result is substituted into the first one, with the result:

$$z[P\tilde{\mathscr{U}}(z) - P] = \left(P\mathscr{U}P + P\mathscr{U}Q\,\frac{1}{z - Q\mathscr{U}Q}\,Q\mathscr{U}P\right) P\tilde{\mathscr{U}}(z) + P\mathscr{U}Q\,\frac{z}{z - Q\mathscr{U}Q}\,Q \tag{33}$$

We now return to the τ-representation. The inverse Z-transforms of the last two terms of Eq. (33) are obtained by expanding the operator $[z - Q\mathscr{U}Q]^{-1}$ in powers of $Q\mathscr{U}Q$ and using Eq. (29). The following operators are now defined:

Propagation fragment:

$$Q\mathscr{P}(\tau)\, Q = (Q\mathscr{U}Q)^{\tau-1}, \qquad \tau \geqslant 1 \tag{34}$$

Diagonal fragment:

$$P\mathscr{E}(\tau)\, P = P\mathscr{U}Q\mathscr{P}(\tau)\, Q\mathscr{U}P, \qquad \tau \geqslant 1 \tag{35}$$

Creation fragment:

$$Q\mathscr{C}(\tau)\, P = Q\mathscr{P}(\tau)\, Q\mathscr{U}P, \qquad \tau \geqslant 1 \tag{36}$$

Destruction fragment:

$$P\mathscr{D}(\tau)\, Q = P\mathscr{U}Q\mathscr{P}(\tau)\, Q, \qquad \tau \geqslant 1 \tag{37}$$

These equations are completed with the following additional ones:

$$Q\mathscr{P}(0)\, Q = P\mathscr{E}(0)\, P = Q\mathscr{C}(0)\, P = P\mathscr{D}(0)\, Q = 0 \tag{38}$$

Note also the simple relation:

$$Q\mathscr{P}(1)\, Q = Q \tag{39}$$

A short calculation, using the shifting theorem (30) and the convolution theorem (31), transforms Eq. (33) into:

$$P\mathscr{U}^{\tau+1} = P\mathscr{U}P\mathscr{U}^{\tau} + \sum_{\sigma=1}^{\tau} P\mathscr{E}(\sigma)\, P\mathscr{U}^{\tau-\sigma} + P\mathscr{D}(\tau+1)\, Q \tag{40}$$

Multiplying this equation to the right by the initial distribution function $f(0)$ and using the (Fourier-transformed) Eq. (16), we find:

$$Pf(\tau+1) = P\mathscr{U}Pf(\tau) + \sum_{\sigma=1}^{\tau} P\mathscr{E}(\sigma)\, Pf(\tau-\sigma) + P\mathscr{D}(\tau+1)\, Qf(0) \tag{41}$$

This is the *discrete-time Master equation*, first derived by Bandtlow and Coveney:$^{(15)}$ it is precisely analogous to the continuous-time Master equation (9). It is a closed equation for the density profile $Pf(\tau) \equiv \varphi(\tau)$; its non-Markovian character is again manifest in the convolution of the second term. It also contains a source term (destruction term) involving the *initial* fluctuations $Qf(0) \equiv g(0)$.

In order to complete the picture, we derive an equation for the fluctuation component of the distribution function (not considered in ref. 15). Solving the second equation (32) we find:

$$Q\widetilde{\mathscr{U}}(z) = Q\, \frac{1}{z - Q\mathscr{U}Q}\, Q\mathscr{U}P\widetilde{\mathscr{U}}(z) + zQ\, \frac{1}{z - Q\mathscr{U}Q}\, Q$$

Right-multiplication of both sides by the initial distribution function $f(0)$ and inverse Z-transformation yields:

$$Qf(\tau) = \sum_{\sigma=0}^{\tau} Q\mathscr{C}(\sigma)\, Pf(\tau-\sigma) + Q\mathscr{P}(\tau+1)\, Qf(0), \qquad \tau \geqslant 1 \tag{42}$$

Unlike Eq. (41), Eq. (42) is an *explicit* relation determining the fluctuation at time τ once the density profile $Pf(\tau)$ is known *at all previous times* from the solution of the Master equation. In other words, *the fluctuation is a "non-Markovian" functional of the density profile*. The structure of this equation is exactly analogous to Eq. (12) in continuous time.

Having completed the derivation of these evolution equations by the method of projections, we simplify the notations, by omitting the writing of the projection operators and using the notations of Eqs. (18) and (24); we also group together the first two terms in the right hand side of Eq. (41) into the *memory kernel* $\psi(\tau)$. Thus:

$$\begin{gathered} Pf(\tau) \to \varphi(\tau), \quad Qf(\tau) \to g(\tau) \\ Q\mathscr{P}(\tau)\, Q \to \mathscr{P}(\tau), \quad Q\mathscr{C}(\tau)\, P \to \mathscr{C}(\tau), \quad P\mathscr{D}(\tau)\, Q \to \mathscr{D}(\tau) \\ P\mathscr{U}P\delta_{\tau,0} + P\mathscr{E}(\tau)\, P \to \psi(\tau) \end{gathered} \tag{43}$$

Equations (41) and (42) are then rewritten in the more compact form:

$$\varphi(\tau+1) = \sum_{\sigma=0}^{\tau} \psi(\sigma)\, \varphi(\tau-\sigma) + \mathscr{D}(\tau+1)\, g(0) \tag{44}$$

$$g(\tau) = \sum_{\sigma=0}^{\tau} \mathscr{C}(\sigma)\, \varphi(\tau-\sigma) + \mathscr{P}(\tau+1)\, g(0) \tag{45}$$

Let us recall that Eq. (45) is actually an abbreviation for a set of individual equations for the Fourier components $g_m(q; \tau)$, $m = \pm 1, \pm 2,...$ Eq. (39) ensures that for $\tau = 0$ Eq. (45) reduces to the identity $g(0) = g(0)$.

As can be seen from the derivation, no approximation has been made: Eqs. (44) and (45) are *exact equations*.[3] Let us now assume that there exist two widely separated time scales. The short *memory time* τ_M characterizes the decay of the four operators defined in Eqs. (34)–(37)

$$\psi(\tau) \approx \mathscr{D}(\tau) \approx \mathscr{C}(\tau) \approx \mathscr{P}(\tau) \approx 0, \qquad \tau > \tau_M \tag{46}$$

The long *relaxation time* τ_R measures the rate of change of the density profile; it will be precisely defined in Section 7. In this case we may perform an asymptotic *markovianization*, just as in Section 2. The last term in Eqs. (44) and (45) is neglected, the retardation is neglected in $\varphi(\tau-\sigma)$, and the upper limit in the summations over σ is extended to infinity. The resulting equations are:

$$\varphi(\tau+1) = \Psi\varphi(\tau), \qquad \tau \gg \tau_M \tag{47}$$

$$g(\tau) = \mathbb{C}\varphi(\tau), \qquad \tau \gg \tau_M \tag{48}$$

---

[3] The only assumptions necessary for their validity are technical assumptions about the smoothness of the functions involved here.(15) We do not discuss these mathematical aspects here.

The time-independent operators appearing in these *Markovian kinetic equations* are:

$$\Psi = \sum_{\sigma=0}^{\infty} \psi(\sigma) \tag{49}$$

$$\mathbb{C} = \sum_{\sigma=0}^{\infty} \mathscr{C}(\sigma) \tag{50}$$

It is worth stressing the formal analogy between the initial Perron–Frobenius equation (15) and the Markovian kinetic equation (47): in both cases the unknown function at time $\tau+1$ is expressed by the action of a time-independent operator on its value at time τ. The essential difference between the two equations is that the former involves the complete distribution function (i.e., the infinite set of its Fourier components), whereas the latter involves solely the $m=0$ component, i.e., the density profile. All the other Fourier components (i.e., the fluctuations) are determined by the density profile through the explicit relation (48). Here resides the essential reduction of the description achieved by the asymptotic kinetic regime, whenever it is valid.(18)

The exact non-Markovian Master equations (44) and (45), and their asymptotic Markovian approximations (47) and (48) constitute the basis of the kinetic theory of area-preserving maps. They will now be applied to the case of the standard map.

4. KINETIC EQUATIONS FOR THE STANDARD MAP

The advantage of the standard map lies in the fact that it is, on one hand a useful model for many real physical problems, and on the other hand it is sufficiently simple for performing many calculations explicitly. In particular, the Perron–Frobenius operator, which requires the inversion of the map (2), can be written explicitly as follows in the (x, θ) representation:(6, 14)

$$\mathscr{U} = \exp\left(-x\frac{\partial}{\partial\theta}\right)\exp\left(\frac{K}{2\pi}\sin 2\pi\theta\frac{\partial}{\partial x}\right) \tag{51}$$

The Perron–Frobenius operator clearly depends on the stochasticity parameter K; for brevity, this dependence is not written down explicitly in the present section. The result of its action on the distribution function is:

$$\begin{aligned} F[x, \theta; \tau+1] &= \mathscr{U}F[x, \theta; \tau] \\ &= F\left[x + \frac{K}{2\pi}\sin 2\pi(\theta - x), \theta - x; \tau\right] \end{aligned} \tag{52}$$

We now go over to the Fourier representation defined by Eq. (17). The action of the Perron–Frobenius operator is then written as follows:

$$f_M(q;\tau+1)=\sum_{M'=-\infty}^{\infty}\int_{-\infty}^{\infty} dq'\langle q, M|\, \mathscr{U}\, |q', M'\rangle\, f_{M'}(q';\tau) \tag{53}$$

The matrix element of the Perron–Frobenius operator is defined as usual (e.g., in quantum mechanics):

$$\begin{aligned}\langle q, M|\, \mathscr{U}\, |q', M'\rangle &=\int_{-\infty}^{\infty} dx\int_0^1 d\theta\, \exp[-2\pi i(qx+M\theta)]\\ &\quad\times \exp\left\{2\pi i\left[q'x+q'\frac{K}{2\pi}\sin 2\pi(\theta-x)+M'(\theta-x)\right]\right\}\\ &=\sum_N\int_{-\infty}^{\infty} dx\int_0^1 d\theta\, \exp[2\pi i(-q+q'-N-M')\,x]\\ &\quad\times\exp[2\pi i(-M+M'+N)\,\theta]\, J_N(q'K)\end{aligned} \tag{54}$$

The well-known identity for the Bessel function of order N, $J_N(z)$ has been used:

$$\exp(iz\sin\phi)=\sum_{N=-\infty}^{\infty} e^{iN\phi}J_N(z) \tag{55}$$

The integrations in Eq. (54) are now easily done, with the result:

$$\langle q, M|\, \mathscr{U}\, |q', M'\rangle=\delta(q'-q-M)\, J_{M-M'}(q'K) \tag{56}$$

The basic operators entering the Master equations are easily obtained in the Fourier representation. The calculation is sketched in Appendix A. They are most compactly expressed in terms of the following operator:

$$\begin{aligned}\hat{\mathscr{W}}_{M_0, M_f}(q\mid q';\tau)&=\sum_{m_1}\cdots\sum_{m_\tau}\delta\left(q'-q-M_0-\sum_{j=1}^{\tau} m_j\right)\\ &\quad\times J_{M_0-m_1}[(q+M_0)\,K]\, J_{m_1-m_2}[(q+M_0+m_1)\,K]\\ &\quad\times J_{m_2-m_3}[(q+M_0+m_1+m_2)\,K]\cdots J_{m_\tau-M_f}(q'K),\quad \tau\geqslant 1\end{aligned} \tag{57}$$

According to our convention, in all sums over (lower case) m_j the value 0 is excluded. We use the convention of writing a hat over the symbols denoting "double matrix elements", i.e., objects depending both on a couple of integers (M_0, M_f) and on a couple of (continuous) wave vectors (q, q').

The Master equation for the density profile (44) is now written as follows:

$$\varphi(q; \tau+1) = \sum_{\sigma=0}^{\tau} \int dq'\, \hat{\psi}(q \mid q'; \sigma)\, \varphi(q'; \tau-\sigma) + \sum_{m'} \int dq'\, \hat{\mathscr{D}}_{m'}(q \mid q'; \tau+1)\, g_{m'}(q'; 0) \tag{58}$$

The *memory kernel* is obtained as:

$$\begin{aligned} \hat{\psi}(q \mid q'; 0) &= \delta(q'-q)\, J_0(qK) \\ \hat{\psi}(q \mid q'; \sigma) &= \hat{\mathscr{W}}_{0,0}(q \mid q'; \sigma), \qquad \sigma \geqslant 1 \end{aligned} \tag{59}$$

The *destruction operator* is:

$$\begin{aligned} \hat{\mathscr{D}}_{m'}(q \mid q'; 1) &= \delta(q'-q)\, J_{-m'}(qK) \\ \hat{\mathscr{D}}_{m'}(q \mid q'; \tau+1) &= \hat{\mathscr{W}}_{0,m'}(q \mid q'; \tau), \qquad \tau \geqslant 1 \end{aligned} \tag{60}$$

We now consider the equation for the fluctuations, (45), which is written as:

$$g_m(q; \tau) = \sum_{\sigma=0}^{\tau} \int dq'\, \hat{\mathscr{C}}_m(q \mid q'; \sigma)\, \varphi(q'; \tau-\sigma) + \sum_{m'} \int dq'\, \hat{\mathscr{P}}_{m,m'}(q \mid q'; \tau+1)\, g_{m'}(q'; 0) \tag{61}$$

The *creation operator* is:

$$\begin{aligned} \hat{\mathscr{C}}_m(q \mid q'; 0) &= 0 \\ \hat{\mathscr{C}}_m(q \mid q'; 1) &= \delta(q'-q-m)\, J_m[(q+m)\, K] \\ \hat{\mathscr{C}}_m(q \mid q'; \sigma) &= \hat{\mathscr{W}}_{m,0}(q \mid q'; \sigma-1), \qquad \sigma \geqslant 2 \end{aligned} \tag{62}$$

The *propagation operator* is:

$$\begin{aligned} \hat{\mathscr{P}}_{m,m'}(q \mid q'; 1) &= \delta(q'-q)\, \delta_{m,m'} \\ \hat{\mathscr{P}}_{m,m'}(q \mid q'; 2) &= \delta(q'-q-m)\, J_{m-m'}[(q+m)\, K] \\ \hat{\mathscr{P}}_{m,m'}(q \mid q'; \tau+1) &= \hat{\mathscr{W}}_{m,m'}(q \mid q'; \tau-1), \qquad \tau \geqslant 2 \end{aligned} \tag{63}$$

We stress the fact that the equations of evolution (58)–(63) are *exact equations for the standard map*: no approximation whatever was done in their derivation. We summarize again here their structure.

The basic equation is the Master equation (58), a *closed equation for the density profile*. It is a *non-Markovian* equation in discrete time, as the unknown function is involved in a convolution in τ; it is, moreover an *integral equation* with respect to the wave-vector q. Finally, we note the presence of the destruction fragment which is, mathematically, a source term, involving the initial value of the correlations. These features make the equation rather complicated. Let us add the fact that the operator $\hat{\mathscr{W}}$ defined in Eq. (57) involves multiple series in m, and we shall be convinced that there is no hope for an *exact* solution of these equations. It would, indeed, have been surprising if there existed a simple general solution of the Master equation, given the known complexity of the standard map orbits for $K > 1$.

Equation (61) for the fluctuations looks very similar to the master equation (58). There is, however, a very important difference: the former is not an equation at all. It rather expresses explicitly the unknown function $g_m(q; \tau)$ in terms of the density profile and of the initial fluctuations. If a solution of the master equation has been obtained in the first step, the fluctuations are obtained from eq. (61) by a mere quadrature and a summation over the past values of the density profile, together with a contribution of the initial fluctuations.

5. THE DIFFUSIVE REGIME

In order to obtain specific results, it is necessary to look for special regimes, i.e., ranges of parameters and of variables, for which the general equations become simpler. It has been known since many years(2–4, 7–13) that in the limit of large stochasticity parameter K the dynamics of the standard map tends toward a diffusive dynamics, characterized by a diffusion coefficient $D(K)$. It has later been shown by Hasegawa and Saphir(6, 14) that the condition of large K must be coupled to a condition of small wave vector q, i.e., of large spatial scales, in order to reach a truly diffusive regime. More precisely, they defined the following range for the diffusive regime:

$$\sqrt{K} \gg 1, \qquad qK \ll 1 \tag{64}$$

The second condition is, however, not expressed in a satisfactory way, as it involves a variable (q) rather than an external parameter. The more precise formulation must be done as follows. We assume that both the density profile and the fluctuations are initially sharply peaked around $q = 0$, with a width much smaller than K^{-1}. The conditions (64) will thus be replaced by the following (somewhat more stringent) ones:[4]

$$\begin{aligned} &\sqrt{K} \gg 1 \\ &\varphi(q, K; 0) \approx 0, \qquad |q| \gtrsim K^{-1} \\ &g_m(q, K; 0) \approx 0, \qquad |q| \gtrsim K^{-1}, \quad \forall m \end{aligned} \tag{65}$$

In order for the forthcoming approximations to hold, it must be checked *a posteriori* that the standard map dynamics ensures the validity of these relations at *all positive times*:

$$\begin{aligned} &\varphi(q, K; \tau) \approx 0, \qquad |q| \gtrsim K^{-1}, \quad \tau > 0 \\ &g_m(q, K; \tau) \approx 0, \qquad |q| \gtrsim K^{-1}, \quad \forall m, \tau > 0 \end{aligned} \tag{66}$$

A direct consequence of the sharp peaking of the distribution functions is the following:

$$\begin{aligned} &\varphi[q + n, K; \tau] \approx 0, \qquad |q| \ll 1, \quad |n| = \pm 1, \pm 2, \ldots \\ &g_m[q + n, K; \tau] \approx 0, \qquad |q| \ll 1, \quad |n| = \pm 1, \pm 2, \ldots, \quad \forall m \end{aligned} \tag{67}$$

We now return to Eqs. (58) and (61): in both of these the value of a distribution function for wave vector q is related in the right hand side to a distribution function at a different value q'. As appears from the definition (57), the latter differs from the former by a certain *integer*: $M_0 + \sum_{j=1}^{\tau} m_j$. It then follows from Eq. (67) that the corresponding contributions are negligibly small, except when this integer is zero. Hence, in the diffusive regime we only have to retain terms in the summations where this condition is satisfied. This then implies that the δ-function in (57) reduces to $\delta(q' - q)$, i.e., all the matrix operators reduce to their diagonal matrix elements (in q), and the integrations over q' in Eqs. (58) and (61) can be trivially performed. We thus introduce the following new definition of "hatless" matrix elements [which depend on a single couple of integer wave-numbers (M_0, M_f)]:[5]

---

[4] From here on, the K-dependence of the various quantities will be indicated explicitly.

[5] We use here the more convenient notation for the Kronecker delta symbol:

$$\delta_{Kr}(M_0 + m_1 + \cdots + m_\tau) = \delta_{M_0 + m_1 + \cdots + m_\tau, 0}$$

$$\mathscr{W}_{M_0,M_f}(q,K;\tau)=\sum_{m_1}\sum_{m_\tau}\delta_{Kr}(M_0+m_1+\cdots+m_\tau)\times J_{M_0-m_1}[(q+M_0)K]\,J_{m_1-m_2}[(q+M_0+m_1)K]\times J_{m_2-m_3}[(q+M_0+m_1+m_2)K]\cdots J_{m_\tau-M_f}(qK),\quad \tau\geqslant 1 \tag{68}$$

The *Master equation in the diffusive regime* reduces to:

$$\varphi(q,K;\tau+1)=\sum_{\sigma=0}^{\tau}\psi(q,K;\sigma)\,\varphi(q,K;\tau-\sigma)+\sum_{m'}\mathscr{D}_{m'}(q,K;\tau+1)\,g_{m'}(q;0) \tag{69}$$

The *memory kernel* is now:

$$\begin{aligned}\psi(q,K;0)&=J_0(qK)\\ \psi(q,K;1)&=0\\ \psi(q,K;\sigma)&=\mathscr{W}_{0,0}(q,K;\sigma),\qquad \sigma\geqslant 2\end{aligned} \tag{70}$$

The *destruction operator* is:

$$\begin{aligned}\mathscr{D}_{m'}(q,K;1)&=J_{-m'}(qK)\\ \mathscr{D}_{m'}(q,K;2)&=0\\ \mathscr{D}_{m'}(q,K;\tau+1)&=\mathscr{W}_{0,m'}(q,K;\tau),\qquad \tau\geqslant 2\end{aligned} \tag{71}$$

The *equation for the fluctuations* reduces to:

$$g_m(q,K;\tau)=\sum_{\sigma=0}^{\tau}\mathscr{C}_m(q,K;\sigma)\,\varphi(q,K;\tau-\sigma)+\sum_{m'}\mathscr{P}_{m,m'}(q,K;\tau+1)\,g_{m'}(q;0) \tag{72}$$

The *creation operator* is:

$$\begin{aligned}\mathscr{C}_m(q,K;0)&=0\\ \mathscr{C}_m(q,K;1)&=0\\ \mathscr{C}_m(q,K;\sigma)&=\mathscr{W}_{m,0}(q,K;\sigma-1),\qquad \sigma\geqslant 2\end{aligned} \tag{73}$$

The *propagation operator* is:

$$\begin{aligned}\mathscr{P}_{m,m'}(q,K;1)&=\delta_{m,m'}\\ \mathscr{P}_{m,m'}(q,K;2)&=0\\ \mathscr{P}_{m,m'}(q,K;\tau+1)&=\mathscr{W}_{m,m'}(q,K;\tau-1),\qquad \tau\geqslant 2\end{aligned}\tag{74}$$

Eqs. (69) and (72) are the basic *Master equations for the standard map in the diffusive regime*. We have already achieved an important simplification at this stage: these are no longer integral equations with respect to the variable q. All the operators reduce to their diagonal elements in q; the equations interrelate only the unknown functions for the same value of the wave vector.[6]

The Master equations remain, however, non-Markovian as they involve convolutions in the (discrete) time τ. An important additional simplification would be justified if it could be shown that the operators $\psi(\tau)$, $\mathscr{D}(\tau)$, $\mathscr{C}(\tau)$, $\mathscr{P}(\tau)$ all decay rapidly, on a time scale τ_M, the memory time. Then, as shown in Section 3, the master equations can be markovianized and reduced asymptotically, for times $\tau\gg\tau_M$, to:

$$\varphi(q,K;\tau+1)=\Psi(q,K)\,\varphi(q,K;\tau)\tag{75}$$

$$g_m(q,K;\tau)=\mathbb{C}_m(q,K)\,\varphi(q,K;\tau)\tag{76}$$

with the following definitions for the time-independent operators:

$$\Psi(q,K)=\sum_{\sigma=0}^{\infty}\psi(q,K;\sigma)\tag{77}$$

$$\mathbb{C}_m(q,K)=\sum_{\sigma=0}^{\infty}\mathscr{C}_m(q,K;\sigma)\tag{78}$$

6. THE MEMORY KERNEL IN THE DIFFUSIVE REGIME

Even in the diffusive regime, defined by Eqs. (69) for the density profile, and (72) for the fluctuations, the basic operators cannot be

---

[6] It may be noted that Hasegawa and Saphir$^{(6,14)}$ used from the very beginning a definition of the projection operator as $P_q=|q,0\rangle\langle q,0|$ for a *given*, *fixed* value of q. This implies that they consider a state $\varphi(q';\tau)$ (for $q'\neq 0$) as a fluctuation, which is quite unnatural. The definition used here comprises in the P-states the set of states $f_{M=0}(q;\tau)$ for *all* values of q. The diagonalization of the operators appears as an approximation valid in the diffusion regime.

evaluated exactly, because for $\tau \geqslant 2$ they involve multiple infinite summations over wave numbers m_j. We must therefore, in practice, introduce approximations which produce the truncation of these series after a finite number of terms. These approximations are validated by the very definition of the diffusive regime, Eq. (65), or (64).

We first consider the *memory kernel* $\psi(q, K; \tau)$, defined by Eq. (70) which, combined with (68) (with $M_0 = M_f = 0$) yields:

$$\begin{aligned}\psi(q, K, \tau) = \sum_{m_1} \cdots \sum_{m_\tau} & \delta_{Kr}(m_1 + \cdots + m_\tau) \\ & \times J_{-m_1}(qK)\, J_{m_1 - m_2}[(q + m_1)\, K] \\ & \times J_{m_2 - m_3}[(q + m_1 + m_2)\, K] \cdots J_{m_\tau}(qK) \end{aligned} \tag{79}$$

for $\tau \geqslant 2$. This equation involves two types of Bessel functions.

- The two extreme (right and left) factors are of the type $J_m(qK)$, where the argument $qK \ll 1$, as follows from Eq. (64). It is well known that, to dominant order in the small argument, the Bessel function behaves as:

$$J_m(qK) \sim \left(\frac{qK}{2}\right)^{|m|} \tag{80}$$

- The intermediate factors are of the type $J_m(qK + nK)$, where n is some non-zero integer. In this case, because of (64), $nK \gg qK$, and the order of magnitude of the factor is found from the asymptotic limit of the Bessel function:

$$\begin{aligned} J_m(qK + nK) & \approx J_m(nK) \\ & \sim \sqrt{\frac{2}{\pi}}\, (nK)^{-1/2} \cos(nK - ((2m+1)/4)\, \pi) \end{aligned} \tag{81}$$

As a result, the general term of the series in Eq. (79) is of order:[7]

$$(qK)^{|m_1| + |m_\tau|}\, K^{-(\tau-1)/2} \tag{82}$$

This first, crude estimate shows that the memory kernel $\psi(q, K; \tau)$ decreases in time roughly as $K^{-\tau/2}$; the various terms in its defining series are, however, of different orders of magnitude with respect to the two small

---

[7] Note that in a few of these terms n may be zero. Any such term containing an intermediate factor with $n = 0$ produces an additional power of qK and one less power of $K^{-1/2}$.

parameters (qK) and $K^{-1/2}$. Moreover, the estimate (82) only provides a kind of "envelope": the precise dependence on K is complicated by the oscillations of the Bessel functions and their relative phase relationships [see (81)].

In order to calculate approximately $\psi(q, K; \tau)$, certain criteria must be defined, allowing the truncation of the multiple series in Eq. (79). The first, and most important criterion is the order in (qK). To a given order in this quantity, there is still, however, an infinite number of terms (for $\tau \geqslant 4$). The criterion for the truncation of this subseries is explained in Appendix B. The following expressions are obtained through order $(qK)^4$:

$$\psi(q, K; 0) = J_0(qK) \tag{83}$$

$$\psi(q, K; 2) = J_1^2(qK)[J_2(K+qK) + J_2(K-qK)] + 2J_2^2(qK)\, J_4(2K) \tag{84}$$

$$\begin{aligned}\psi(q, K; 3) = &-2J_1^2(qK)\, J_3(K-qK)\, J_3(K+qK)\\ &-2J_1(qK)\, J_2(qK)[J_5(2K-qK)\, J_4(K+qK)\\ &-J_5(2K+qK)\, J_4(K-qK) + J_0(K+qK)\, J_3(2K+qK)\\ &-J_0(K-qK)\, J_3(2K-qK)] + 2J_2^2(qK)\, J_6^2(2K)\\ &-4J_1(qK)\, J_3(qK)[J_5(K)\, J_7(3K) + J_1(K)\, J_5(3K)]\end{aligned} \tag{85}$$

As explained in Appendix B, the expression of $\psi(q, K; 4)$ is very long, and will not be written down here; it was, however retained in the forthcoming calculations.

It is immediately seen that $\psi(q, K; 0)$ very strongly dominates the values for positive τ. The reason for this clearly appears from Eqs. (83)–(85): $\psi(q, K; 0) = J_0(qK) \sim 1$, whereas for all other values of τ the memory kernel is of dominant order $J_1^2(qK) \sim (qK)^2 \ll 1$.

The memory kernel is plotted in Fig. 1 as a function of q, for fixed K and for $\tau = 2, 3, 4$. The range chosen for q is $(0, 0.02)$; the value chosen for K in all forthcoming figures at fixed K will be $K = 22.5$; we thus obtain: $qK \leqslant 0.45$, $\sqrt{K} = 4.74$. Both values are somewhat marginal for the criterion (64); the value of q is an upper limit, the bulk of the density profile chosen below will be in the range below 0.005. On the other hand, too large a value of K, while favorable for the mathematics, would not be very interesting for physical applications (it would, in particular, require an even smaller maximum q).

Clearly, $\psi(q, K; 2)$ dominates, but $|\psi(q, K; 3)| < |\psi(q, K; 4)|$. This "irregularity" will be understood from the discussion that follows. Meanwhile, we note that, even for the largest q, the value of $|\psi(q, K; \tau)|$

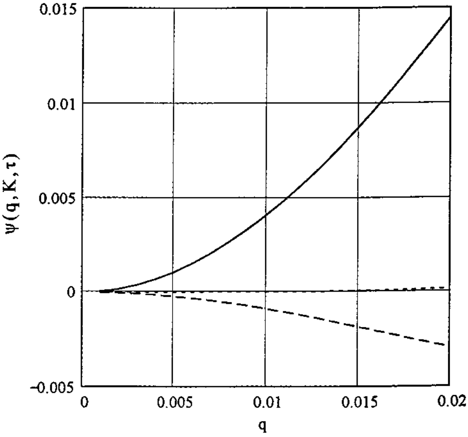

Fig. 1. The memory kernel $\psi(q, K; \tau)$ for fixed $K = 22.5$, vs. the wave vector q. Solid: $\tau = 2$; dotted: $\tau = 3$; dashed: $\tau = 4$. $\psi(q, K; 0) = J_0(qK) \approx 1$ is not represented.

has dropped from 1 (for $\tau = 0$) to about 0.003 for $\tau = 4$. The sum in the convolution of the right hand side of Eq. (69) [or (87) below] can therefore be truncated at $\sigma = 4$. This property leads to a natural definition of the *memory time* τ_M mentioned defined in Eq. (46):

$$|\psi(q, K; \tau)| \approx 0 \qquad \text{for} \quad \tau > \tau_M \tag{86}$$

In the case considered here we find: $\tau_M \approx 4$.[8]

We now consider the memory kernel for fixed $q = 0.005$ (in the bulk of the density profile) as a function of the stochasticity parameter (Fig. 2). The most striking feature is the *oscillatory nature of the memory kernel as a function of* K. The period of the oscillations is approximately constant ($\Delta K_2 \approx 6.4$ for $\psi(q, K; 2)$, $\Delta K_3 \approx \Delta K_4 \approx \frac{1}{2} \Delta K_2$), but the amplitudes are slowly increasing with K for all three functions. Note that $\psi(q, K; 4)$ is approximately in phase with $\psi(q, K; 2)$, whereas $\psi(q, K; 3)$ is out of phase with the former two. More precisely, the zeroes of $\psi(q, K; 2)$ approximately coincide with zeroes of $\psi(q, K; 4)$ and with (negative) minima of $\psi(q, K; 3)$. On the other hand, the extrema of $\psi(q, K; 2)$ approximately coincide with positive maxima of $\psi(q, K; 4)$ and with zeroes (i.e., maxima) of $\psi(q, K; 3)$. For our default value $K = 22.5$, both $\psi(q, K; 2)$ and $\psi(q, K; 4)$ have a positive maximum, whereas $\psi(q, K; 3)$ is almost zero: this corresponds indeed with Fig. 1. We finally note that in the range of K represented in

---

[8] The memory time τ_M cannot be defined sharply, but only as an order of magnitude. Its value depends on the degree of accuracy accepted in the theory.

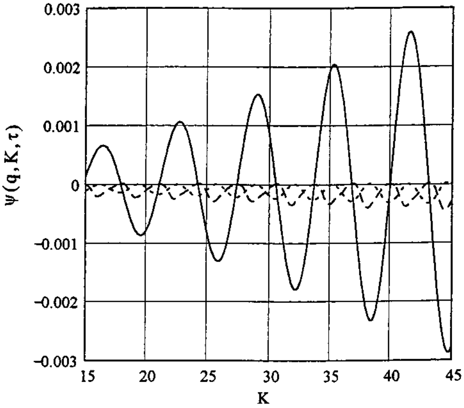

Fig. 2. The memory kernel $\psi(q, K; \tau)$ for fixed $q = 0.005$, vs. the stochasticity parameter K. Solid: $\tau = 2$; dotted: $\tau = 3$; dashed: $\tau = 4$.

Fig. 2, the function $\psi(q, K; 0)$ does not oscillate: it decreases extremely slowly and monotonously (varying from 0.999 at $K = 15$ to 0.987 at $K = 45$).

7. SOLUTION OF THE MASTER EQUATION FOR THE DENSITY PROFILE

The master equation (70) will now be analyzed in two successive steps. We first assume that the *initial condition is a density profile, independent of the angle*; in Fourier space this amounts to taking: $\varphi(q, K; 0) = \varphi_0(q)$, $g_m(q, K; 0) = 0$, $\forall m$. The second (destruction) term in the right hand side of Eq. (69) is then identically zero. Moreover, given the rapid decay of the memory kernel in time, we decide to *truncate the convolution at* $\sigma = 4$ $(= \tau_M)$ (except, of course, for $\tau < 4$). The master equation then reduces to the following simpler form:

$$\varphi(q, K; \tau + 1) = \sum_{\sigma = 0}^{\tau_C} \psi(q, K; \sigma)\, \varphi(q, K, \tau - \sigma)$$
$$\tau_C = \begin{cases} \tau, & \text{for} \quad \tau \leqslant 4 \\ 4, & \text{for} \quad \tau > 4 \end{cases} \tag{87}$$

We also introduce the propagator $H(q, K; \tau)$ relating the density profile at time τ to its initial value $\varphi_0(q)$:

$$\varphi(q, K, \tau) = H(q, K; \tau)\, \varphi_0(q) \tag{88}$$

this function obeys the same equation (87), with an obvious initial condition:

$$\begin{aligned} H(q,K;\tau+1) &= \sum_{\sigma=0}^{\tau_C} \psi(q,K;\sigma)\, H(q,K;\tau-\sigma) \\ H(q,K;0) &= 1 \end{aligned} \tag{89}$$

This equation can be solved analytically if the following approximation is accepted. Terms of the following orders are retained: arbitrary powers of $\psi(q, K; 0)$, terms of order $\psi(q, K; 2)$ and $[\psi(q, K; 2)]^2$; terms of first order in $\psi(q, K; 3)$ and $\psi(q, K; 4)$. Each of these functions in turn, is evaluated through order $(qK)^4$, as explained above. The derivation of the solution is given in Appendix C. The result is written by using the following abbreviations: $H_\tau \equiv H(q, K; \tau)$, $\psi_\tau \equiv \psi(q, K; \tau)$:

$$\begin{aligned} &H_0 = 1 \\ &H_1 = \psi_0 \\ &H_2 = \psi_0^2 \\ &H_3 = \psi_0^3 + \psi_2 \\ &H_4 = \psi_0^4 + 2\psi_0\psi_2 + \psi_2 \\ &H_5 = \psi_0^5 + 3\psi_0^2\psi_2 + 2\psi_0\psi_3 + \psi_4 \\ &H_\tau = \psi_0^\tau + (\tau-2)\,\psi_0^{\tau-3}\psi_2 + (\tau-3)\,\psi_0^{\tau-4}\psi_3 + (\tau-4)\,\psi_0^{\tau-5}\psi_4 \\ &\qquad + \tfrac{1}{2}(\tau-5)(\tau-6)\,\psi_0^{\tau-6}\psi_2^2, \\ &\quad \tau \geqslant 6 \end{aligned} \tag{90}$$

We now consider the *Markovian approximation* to the master equation. Two cases can be considered. The first, crudest approximation consists of neglecting the memory effects altogether in Eq. (87); this amounts to putting $\psi(q, K; \sigma) = 0$ for all $\sigma > 0$ in Eq. (87), which then reduces to:

$$\varphi_Z(q,K;\tau+1) = \psi(q,K;0)\,\varphi_Z(q,K,\tau), \qquad \text{Zero-Markovian} \tag{91}$$

This approximation will be called "*Zero-Markovian.*" Recalling the general definition (43), (and using again the abbreviated notation) we see that:

$$\psi_0 = P\mathscr{U}P \tag{92}$$

Thus the memory kernel in this approximation is simply the diagonal $(P-P)$ element of the Perron–Frobenius operator. Writing Eq. (91) in the general form (41):

$$Pf(\tau+1)=P\mathscr{U}Pf(\tau) \tag{93}$$

we note the striking analogy with the *Vlasov equation* of continuous-time kinetic theory of gases or plasmas, obtained from Eq. (9):

$$\frac{\partial}{\partial t}VF(t)=V\mathscr{L}VF(t) \tag{94}$$

In both equations the fluctuations—respectively, the correlations—are completely neglected, even in the intermediate "virtual" states of the evolution operator.

The solution of the Zero-Markovian equation (91) is immediate:

$$\varphi_Z(q,K;\tau)=[\psi(q,K;0)]^{\tau}\,\varphi_0(q), \qquad \text{Zero-Markovian} \tag{95}$$

The more accurate Markovianization process described in Section 3, combined with the truncation performed in our problem yields the following form:

$$\varphi_M(q,K;\tau+1)=\Psi(q,K)\,\varphi_M(q,K;\tau), \qquad \text{Full Markovian} \tag{96}$$

with:

$$\Psi(q,K)=\psi(q,K;0)+\psi(q,K;2)+\psi(q,K;3)+\psi(q,K;4) \tag{97}$$

The solution of Eq. (96) is straightforward:

$$\varphi_M(q,K;\tau)=[\Psi(q,K)]^{\tau}\,\varphi_0(q), \qquad \text{Full Markovian} \tag{98}$$

In the following figures we evaluated numerically the solutions defined above. We took an initial condition for the density profile that is exponential in the x-space:

$$n_0(x)=\frac{\lambda}{2}\exp(-\lambda\,|x|) \tag{99}$$

Its Fourier transform is:

$$\varphi_0(q)=\frac{\lambda^2}{\lambda^2+(2\pi q)^2} \tag{100}$$

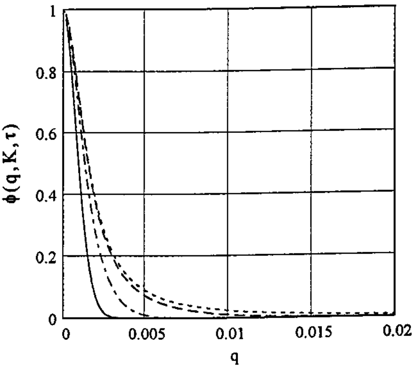

Fig. 3. The density profile $\varphi(q, K; \tau)$, solution of the non-Markovian master equation, vs. τ. $K = 22.5$. Dotted: $\tau = 0$; dashed: $\tau = 100$; dash-dotted: $\tau = 1000$; solid: $\tau = 5000$.

We choose the value $\lambda = 0.01$ for the characteristic wavenumber of this profile. This Lorentzian function is properly normalized in agreement with Eq. (19); it is sharply peaked around $q = 0$ and, for the default value $K = 22.5$, satisfies the criterion (65).

In Fig. 3 the solution $\varphi(q, K; \tau)$, Eq. (88) is plotted as a function of q for $\tau = 0$, 100, 1000, 5000. The overall evolution is a narrowing of the function $\varphi(q, K; \tau)$ for increasing times. In other words, the long tail of the Lorentzian is progressively cut off, whereas the value $\varphi(0, K; \tau) = 1$ is maintained for all times (conservation of the normalization). This evolution shows, *a posteriori*, that the criterion (66) is better and better satisfied as time passes. The narrowing in Fourier space corresponds to a spreading in x-space, hence, grossly speaking, to a dispersive behaviour. Whether this "dispersion" can be called "diffusion" has to be further examined.

We now compare the non-Markovian solution to the two Markovian approximations. Consider first the zero-Markovian solution, which has a simple analytical form. From Eqs. (95) and (83) the following expression is found for the propagator, in the domain of small (qK):

$$H_Z(q, K; \tau) = [J_0(qK)]^\tau \approx \left[1 - \frac{1}{4}(qK)^2\right]^\tau \approx \exp\left(-\frac{1}{4}K^2 q^2 \tau\right) \tag{101}$$

This has precisely the form of the propagator associated with the diffusion equation (in Fourier space), with a diffusion coefficient D:

$$H_Z(q; \tau) \approx \exp[-D(K)(2\pi q)^2 \tau] \tag{102}$$

The diffusion coefficient appearing in the zero-Markovian approximation (101) is thus:

$$D(K)=D_{QL}(K)=\frac{1}{4(2\pi)^2}\,K^2 \tag{103}$$

This is the well-known "*quasilinear diffusion coefficient.*"$^{(2-4,\,6-14)}$ Thus, the "Vlasov" approximation for the standard map in the diffusive regime is equivalent to the quasilinear approximation. The important point is that already at this stage, we see that *for large spatial scales and for long times (ensuring the validity of the asymptotic zero-Markovian approximation) the standard map dynamics reduces to a truly diffusive regime characterized by the propagator (102).*

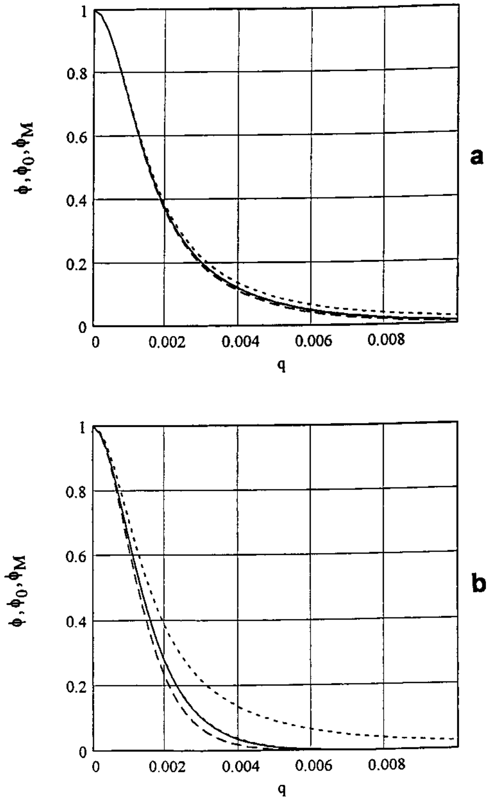

Fig. 4. Non-Markovian compared to the zero-Markovian and full Markovian approximations for the density profile. $K=22.5$. Dotted: initial condition; solid: non-Markovian, superposed to full Markovian; dashed: zero-Markovian. a: $\tau=100$; b: $\tau=1000$.

We now consider the graphs of the non-Markovian, zero-Markovian and full Markovian solutions. For relatively short times the three functions are close together: this is seen in Fig. 4a for $\tau = 100$. For long times the full Markovian solution is very close to the non-Markovian (as expected) for most values of q; *the zero-Markovian "Vlasov" solution, however, is significantly different from the "exact" non-Markovian* (Fig. 4b). Eventually, all three functions will coalesce and become very narrow; but this process will take quite a long time. This is a very interesting result. Recalling Eqs. (75) and (77), it is seen that the memory effect [i.e., $\psi(q, K; \sigma)$ for $\sigma > 0$] cannot be ignored in the markovianization of the evolution equation, i.e., in the construction of the full Markovian memory kernel $\Psi(q, K)$, Eq. (97) and of the propagator $H_M(q, K; \tau) \equiv \exp[\Psi(q, K)\,\tau]$. Thus, the full Markovian approximation should not be understood as a "memoryless" evolution. The evolution operator $\Psi(q, K)$ is built up by the cumulative action of the exact memory kernel over a finite time span of the order of the short memory time τ_M.

The non-Markovian effects are masked in Fig. 4, because of the scale of the figures which encompasses the entire density profile. A clearer view is obtained by plotting in Fig. 5 the difference between the non-Markovian and the full Markovian: $\Delta_{NM}(q, K; \tau) = \varphi(q, K; \tau) - \varphi_M(q, K; \tau)$, for $\tau = 10$, 500, 1000, 5000. The deviation is indeed very small, but it depends on q (and K): as time goes on, it concentrates at small values of q. For $q = 0.005$ it is about ten times larger in absolute value at $\tau = 1000$ than at $\tau = 100$. Percentagewise, however, this peak represents barely about 0.1 %, because it occurs in the bulk of the density profile where $\varphi(q, K; \tau) \approx 1$.

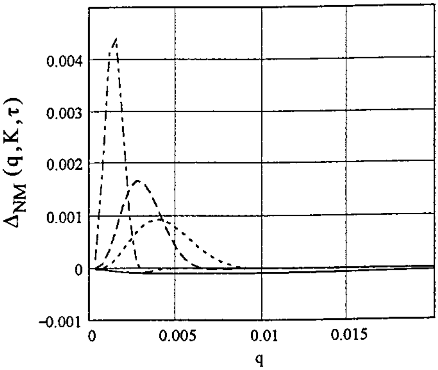

Fig. 5. Deviation of the full Markovian approximation from the non-Markovian solution $\Delta_{NM}(q, K; \tau) = \varphi(q, K; \tau) - \varphi_M(q, K; \tau)$. $K = 22.5$. Solid: $\tau = 10$, dotted: $\tau = 500$; dashed: $\tau = 1000$; dash-dotted: $\tau = 5000$.

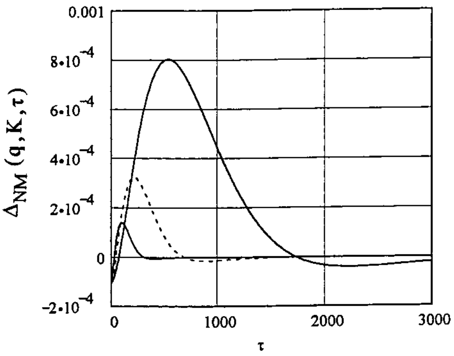

Fig. 6. Deviation of the full Markovian approximation from the non-Markovian solution, $\Delta_{NM}(q, K; \tau)$ vs. τ, for fixed $K = 22.5$, for three values of q. From bottom to top: $q = 0.012$, 0.008, 0.005.

A very important information is obtained by studying the deviation $\Delta_{NM}(q, K; \tau)$ as a function of time (for fixed $K = 22.5$). The graph of Fig. 6 shows the tendency of the non-Markovian toward the Markovian: after an initial growth of the deviation (in absolute value) follows a decay that appears to be exponential. This decay is very slow for small q and very fast in the tail of the distribution. The exponential nature of the decay is clearly apparent by plotting the natural logarithm of $|\Delta_{NM}|$ as a function of time (Fig. 7): after the initial transient regime, this curve is perfectly linear. Upon measuring the slope of the final branch of this curve for several values of q, we find that the result is very accurately fitted (for long times) by the following function:

$$|\Delta_{NM}(q, K; \tau)| = A \exp\left(-\frac{\tau}{\tau_R(q, K)}\right) \tag{104}$$

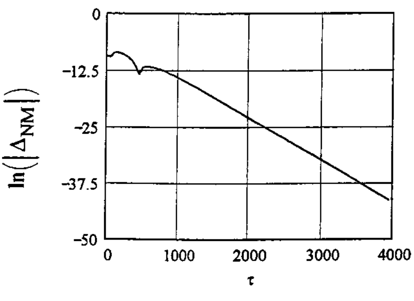

Fig. 7. Natural logarithm of the deviation $|\Delta_{NM}(q, K; \tau)|$ vs. τ for fixed $q = 0.006$ and $K = 22.5$.

where A is an irrelevant constant, and τ_R^{-1} is obtained numerically as:

$$\frac{1}{\tau_R(q, K)} = 92.66q^2, \qquad K = 22.5 \tag{105}$$

The relaxation time $\tau_R(q, K)$ ($\propto q^{-2}$) thus varies rather slowly for large q, but becomes very large for small q, becoming infinite for $q = 0$. Indeed, the value $\varphi(q = 0, K; \tau) = 1$ must remain unchanged in time in order to preserve the normalization.

Two facts are important in this result. First, it offers an *intrinsic and objective definition of the relaxation time* τ_R. In ordinary kinetic theory of gases, an initial non-equilibrium state relaxes exponentially towards a non-trivial, time-independent equilibrium state. This exponential law introduces a relaxation time-scale. In the present case, we found that the asymptotic decay of the deviation is also exponential. Thus, the non-Markovian effects in the "exact" solution decay exponentially after a short transient time. This evolution is characterized by an intrinsic time scale $\tau_R(q, K)$ which we propose to define as *the relaxation time of the non-Markovian (exact) solution evolving towards its asymptotic Markovian regime.*

The second interesting fact is the form (105) of the relaxation time $\tau_R(q, K)$: its dependence on the wave vector is typical of a *diffusion process* [see Eq. (102)]; combining, indeed, Eqs. (104) and (105), the former may be written in the form:

$$|\Delta_{NM}(q, K; \tau)| \sim \exp[-(2\pi q)^2 D_R(K) \tau] \tag{106}$$

with the following value of the diffusion coefficient:

$$D_R(K) = \frac{92.66}{(2\pi)^2} = 2.35, \qquad K = 22.5 \tag{107}$$

This value of the coefficient $D_R(K)$ will be further discussed below. Thus, the relaxation time $\tau_R(q, K)$ is a function of q and K, but is entirely determined by the single scalar function $D_R(K)$ through Eq. (105).

To sum up this discussion, the mechanism of the relaxation of the standard map dynamics in the diffusive regime is characterized by a "double diffusion mechanism". Starting from an arbitrary non-equilibrium state, the system approaches in a non-Markovian way the corresponding Markovian: this evolution is asymptotically diffusive and is characterized by a diffusion coefficient D_R. The Markovian itself relaxes toward spatial homogeneity in an asymptotically diffusive way, controlled by a diffusion coefficient D, that will be calculated below. This "double diffusion

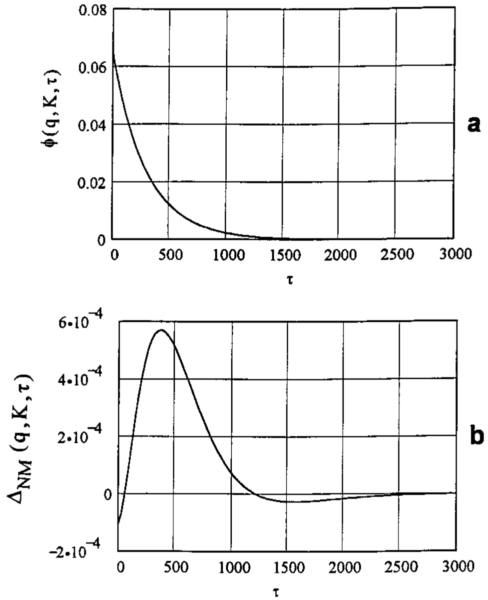

Fig. 8. a: Evolution in time of the non-Markovian solution for the density profile, $\varphi(q, K; \tau)$. b: Evolution in time of the deviation from the non-Markovian solution, $\Delta_{NM}(q, K; \tau)$. $q = 0.006$, $K = 22.5$.

mechanism" is illustrated quantitatively in Fig. 8, in which the evolution in time (for a fixed wave vector $q = 0.006$) of the (non-Markovian) solution and of its deviation from the Markovian are shown in parallel on the same horizontal scale (but not the same vertical scale!). The exponential decay of the former starts after a very short time, whereas the deviation starts its diffusive decay much later (after about $\tau \approx 1500$). As a result, in the long

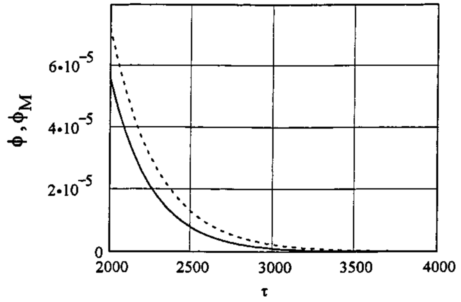

Fig. 9. Long time tails of the non-Markovian, $\varphi(q, K; \tau)$ and Markovian $\varphi_M(q, K; \tau)$ solutions for the density profile. $q = 0.006$, $K = 22.5$.

time tail, where $\varphi(q, K; \tau)$ is very small (say, at $\tau > 2000$), the deviation of the non-Markovian from the Markovian is, percentage-wise, very large. In our example, the deviation at $\tau = 700$ is 17%, whereas for $\tau = 2000$ it represents 200%! This is made visible in a zoom on the long time tails of the non-Markovian and the Markovian solutions (Fig. 9).

8. NON-GAUSSIAN FEATURES OF THE EVOLUTION

If the evolution of the density profile were asymptotically truly diffusive, the latter should tend, for long times, towards a Gaussian packet of the following form (in Fourier space) [see (102)]:[9]

$$\varphi_G(q, K; \tau) = \exp[-\tfrac{1}{2}(2\pi q)^2 \langle x^2(K; \tau)\rangle] \tag{108}$$

where $\langle x^2(K; \tau)\rangle$ is the mean square displacement; for a diffusive regime it is a linear function of time (with a slope depending on K). In the present section we investigate to what extent the non-Markovian and the Markovian solutions approach a Gaussian packet in the long time limit.

We first calculate the mean square displacement, i.e., the second moment of the density profile. Using Eqs. (20) and (88), and the fact that all odd-order derivatives turn out to vanish, we obtain:

$$\langle x^2(K; \tau)\rangle = -\frac{1}{(2\pi)^2} \left\{ \frac{\partial^2 H(q, K; \tau)}{\partial q^2} \varphi_0(q) + H(q, K; \tau) \frac{\partial^2 \varphi_0(q)}{\partial q^2} \right\}\Bigg|_{q=0} \tag{109}$$

Noting that $H(q = 0, K; \tau) = 1$, $\forall \tau$ and $\varphi_0(0) = 1$ [both properties are a result of the normalization, see Eq. (19)], and using Eq. (100), this expression reduces to:

$$\langle x^2(K; \tau)\rangle = -\frac{1}{(2\pi)^2} \frac{\partial^2 H(q, K; \tau)}{\partial q^2}\Bigg|_{q=0} + \frac{2}{\lambda^2} \tag{110}$$

From Eqs. (80) and (83)-(85) it is seen that $\psi(0, K; \tau) = 0$ for $\tau > 0$; it is also checked that $\partial\psi(q; \tau)/\partial q|_{q=0} = 0$ for $\tau \geqslant 0$. It then follows from Eq. (90) that the second derivative of the non-Markovian propagator is:

$$\frac{\partial^2 H(q, K; \tau)}{\partial q^2}\Bigg|_{q=0} = \frac{\partial^2}{\partial q^2} \{\tau\psi_0 + (\tau - 2)\,\psi_2 + (\tau - 3)\,\psi_3 + (\tau - 4)\,\psi\}\,|_{q=0} \tag{111}$$

---

[9] An unfortunate misprint appears in ref. 6. In Eqs. (15.48)–(15.51) and (15.79)–(15.82), as well as in Fig. 15.5, the symbol D must be replaced by $(2\pi)^2 D$ for consistency with the definition (15.41) of the Fourier transform.

This result shows that the mean square displacement is indeed a linear function of time, characteristic of a diffusive regime. It contains a large constant [due to the contribution of the initial condition, the last term of Eq. (110)]. The diffusion coefficient, which depends on K, is defined in the usual way:[10]

$$D(K) = \frac{1}{2} \frac{d\langle x^2(K;\tau)\rangle}{d\tau} \tag{112}$$

We thus find, from (111):

$$D(K) = -\frac{1}{2(2\pi)^2} \sum_{\sigma=0}^{4} \left.\frac{\partial^2 \psi(q, K; \sigma)}{\partial q^4}\right|_{q=0} \tag{113}$$

Before continuing the discussion of this result, we note that the same steps can be repeated using in Eq. (111) the Markovian propagator $H_M(q, K; \tau) = [\Psi(q, K)]^\tau$ defined in Eqs. (97), (98). A short calculation shows that *the diffusion coefficient associated with the Markovian solution exactly equals the non-Markovian one* (113): this is a remarkable result.

The explicit expression of the leading contributions to the diffusion coefficient[11] is:

$$\begin{aligned} D(K) = \frac{K^2}{16\pi^2} \{ &1 - 2J_2(K) + 2J_3^2(K) + 2J_2^2(K) \\ &+ [J_1(K) + J_2(K)]\, J_6(K)\, J_8(K) + 2J_0(K)\, J_4(K)\, J_4(2K) + \cdots \} \end{aligned} \tag{114}$$

This quantity (including the terms not written explicitly) is represented in Fig. 10. The dotted line represents the first term in (114), which is none other than the quasilinear (Vlasov) diffusion coefficient (103). The next terms in Eq. (114) [or (113)] produce oscillations around the quasilinear value. These peculiar oscillations were first discovered in the pioneering work of Rechester and White.(8)

The results obtained so far showed that the second moment $\langle x^2(K;\tau)\rangle$ of the density profile has the characteristic, linearly increasing form of a diffusion process. It is controlled by a single diffusion coefficient that is the same for the "exact" solution and for the asymptotic Markovian one.

---

[10] There is no inconsistency in using the derivative operation for a discrete time in the present case, because $\langle x^2(\tau)\rangle$ is linear in τ.

[11] For brevity, terms containing Bessel functions of the form $J_m(nK)$ with $n > 2$ are not written down in Eq. (114).

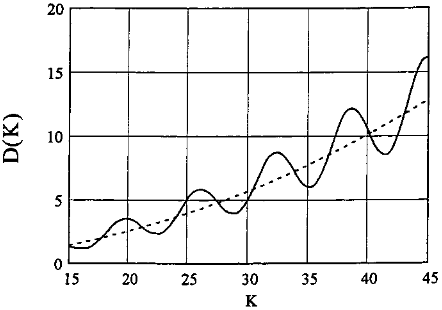

Fig. 10. Diffusion coefficient as a function of K. Solid: Eq. (113); dotted: quasilinear approximation, Eq. (103).

It can also be checked that the numerical value of the diffusion coefficient $D(22.5) = 2.347$ coincides with the coefficient D_R determined by the relaxation time τ_R in Eq. (107). In particular, this implies that *the characteristic time for the relaxation of the deviation from the Markovian is the same as the one determining the asymptotic dynamics of the density profile.*

These features are, however, not sufficient for claiming that the evolution process of the standard map is (asymptotically) diffusive. It must be shown that the density profile approaches, in the long run, the Gaussian packet form (108). A first indication of the deviation of a distribution function from a Gaussian is provided by the *kurtosis*, κ:

$$\kappa(K;\tau) = \frac{\langle x^4(K;\tau)\rangle}{[\langle x^2(K;\tau)\rangle]^2} \tag{115}$$

For a Gaussian distribution this quantity equals 3 for all times.(23, 24) The fourth moment of the non-Markovian solution, related to its fourth derivative by Eq. (20) is:

$$\langle x^4(K;\tau)\rangle = \frac{1}{(2\pi)^4}\left\{\frac{\partial^4 H(q,K;\tau)}{\partial q^4}\varphi_0(q) + 6\,\frac{\partial^2 H(q,K;\tau)}{\partial q^2}\frac{\partial^2\varphi_0(q)}{\partial q^2} + H(q,K;\tau)\,\frac{\partial^2\varphi_0(q)}{\partial q^2}\right\}\bigg|_{q=0} \tag{116}$$

is an exceedingly cumbersome expression. We therefore limited ourselves to a somewhat less accurate approximation, which retains the dominant features: the terms containing $\psi(q,K;4)$ and $[\psi(q,K;2)]^2$ are neglected in Eq. (90).

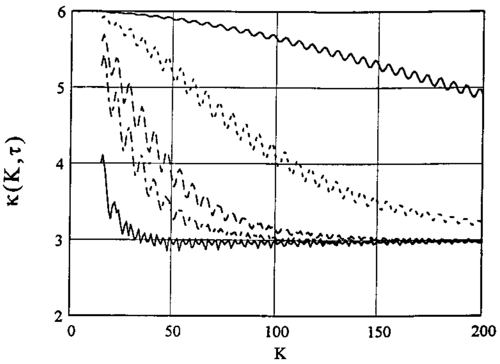

Fig. 11. Kurtosis of the non-Markovian solution for the density profile, vs. K at various times. From top to bottom: $\tau = 10$, 100, 500, 1000, 5000.

The kurtosis of the density profile $\varphi(q, K; \tau)$ is shown in Fig. 11 as a function of K for several times, between 10 and 5000. Note that the kurtosis of the initial density profile (100) equals exactly 6. The kurtosis decreases progressively in time, tending toward the Gaussian value $\kappa = 3$. This decay is faster for the larger values of K. Note again the presence of the ubiquitous oscillations in K. The tendency toward the Gaussian value takes a very long time for moderate values of K: for our default value $K = 22.5$, the kurtosis is still close to 4 after 5000 iterations. Note also that for $K > 50$ and $\tau = 5000$ the kurtosis is very close to 3, but continues to undergo oscillations of significant amplitude in K. The kurtosis for the standard map has been previously calculated by the method of Fourier paths (see Section 11) by Tabet *et al.*$^{(25)}$ with a result qualitatively similar to ours.

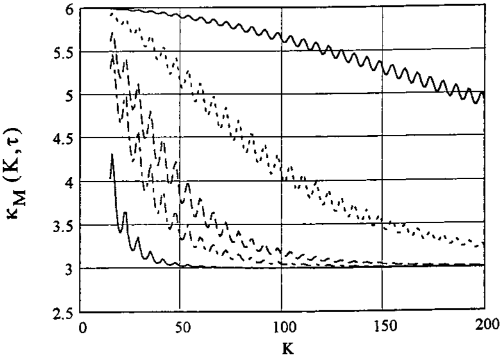

Fig. 12. Kurtosis of the Markovian approximation for the density profile, vs. K at various times. From top to bottom: $\tau = 10$, 100, 500, 1000, 5000.

Figure 12 shows the evolution of the kurtosis for the Markovian solution. The curves are very similar to the previous ones. It may be noted, however, that the oscillations in K disappear completely for $K > 50$ at time $\tau = 5000$: the Markovian in this domain reaches its Gaussian kurtosis value. But for $K = 22.5$, even the Markovian is not yet Gaussian!

A more global view is obtained by comparing the shape of the density profile (88), (90) at various times to the corresponding Gaussian packet (108) evaluated at the same times with the mean square displacement (110). In Fig. 13 we clearly see the density profile "running after the running Gaussian profile", as described above in Section 7. The deviation of

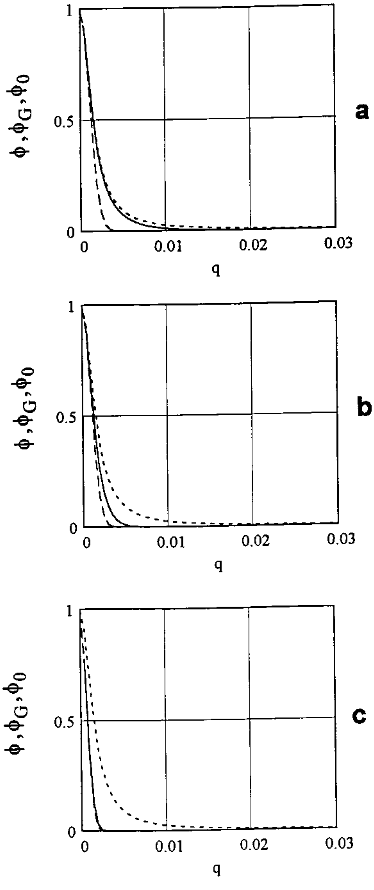

Fig. 13. Comparison of the non-Markovian solution for the density profile $\varphi(q, K; \tau)$ [solid line], the diffusive Gaussian packet $\varphi_G(q, K; \tau)$ [dashed line], and the initial condition, $\varphi_0(q)$ [dotted line]. $K = 22.5$. a: $\tau = 100$, b: $\tau = 1000$, c: $\tau = 7000$.

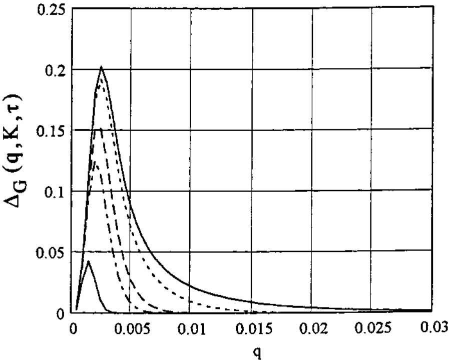

Fig. 14. Deviation of the diffusive Gaussian packet from the non-Markovian solution, $\Delta_G(q, K; \tau) = \varphi(q, K; \tau) - \varphi_G(q, K; \tau)$. $K = 22.5$. From top to bottom: $\tau = 10$, 100, 500, 1000, 5000.

the density profile from the Gaussian, $\Delta_G(q, K; \tau) = \varphi(q, K; \tau) - \varphi_G(q, K; \tau)$ is plotted in Fig. 14 against q for several times. The corresponding graph for the deviation of the Markovian is almost identical.

In conclusion, *the density profile determined by the standard map evolves asymptotically toward a Gaussian packet characteristic of a diffusion process.* Nevertheless, the process takes a long time before the entire density profile adjusts to the Gaussian shape; it does so at different rates for different wave vectors. The time after which φ relaxes to φ_G at various q is of the order of the relaxation time for the same value of q. As a result, even after 5000 iterations, the density profile is not yet exactly Gaussian, especially at small q.

9. THE DESTRUCTION FRAGMENT

In the present section we study the influence of the initial fluctuations on the evolution of the density profile. We therefore consider the complete master equation for the standard map in the diffusive regime, Eq. (69). The general case is, however, very complicated, because the destruction term involves an infinite sum over all Fourier components (with respect to the angle θ). We therefore specialize our study to a particular case, i.e., a particular choice of the initial distribution function (in $x - \theta$ space):

$$F(x, \theta; 0) = n(x; 0)[1 + a \cos 2\pi\mu\theta] \tag{117}$$

where the parameter a is chosen as: $a = 0.5$, and μ is an integer; in the forth-coming calculations its default value is set to $\mu = 2$, unless stated otherwise. It follows that the initial fluctuation in Fourier space is simply:

$$g_m(q;0)=\frac{a}{2}\,\varphi_0(q)[\delta_{m,\mu}+\delta_{m,-\mu}] \tag{118}$$

with $\varphi_0(q)$ defined by Eq. (100). We thus retain a single harmonic in the Fourier series of the initial distribution function, assuming that the latter is an even function of θ.

The general expression of the destruction fragment in the diffusive regime was derived in Eqs. (68) and (71). It appears that for $\tau \geqslant 3$ these expressions become very cumbersome, for the reasons explained in Appendix B. On the other hand, the effect of the destruction fragment is small compared to the memory kernel. We therefore decided to evaluate the former to a lower degree of approximation, retaining only terms through order $(qK)^2$ (which were checked to be clearly dominant). In the factors multiplying the latter for each τ, we chose to neglect terms containing Bessel functions of the form $J_m(nK)$ with $n>2$ (see Appendix B). For brevity, we introduce the following notations:

$$D(q,K;\tau)\equiv D_\tau\equiv\sum_{m'}\mathscr{D}_{m'}(q,K;\tau)\;g_{m'}(q;0) \tag{119}$$

We then find, from Eq. (71), for $\mu=2$:

$$\begin{aligned}
D_1 &= qJ_2(qK)\,\varphi_0(q)\\
D_2 &= 0\\
D_3 &= a[J_0(qK)\,J_2(qK)\,J_4(2K)-J_1^2(qK)\,J_2(K)]\,\varphi_0(q)\\
D_4 &= a\{J_1^2(qK)[J_3^2(K)-J_1(K)\,J_5(3K)-J_5(K)\,J_7(3K)]\\
&\quad+\tfrac{1}{2}J_1(qK)\,J_0(qK)[J_0(K-qK)\,J_3(2K-qK)\\
&\quad-J_0(K+qK)\,J_3(2K+qK)+J_4(K-qK)\,J_5(2K+qK)\\
&\quad-J_4(K+qK)\,J_5(2K-qk)]+J_0(qK)\,J_2(qK)\,J_6^2(2K)\}\;\varphi_0(q)
\end{aligned} \tag{120}$$

We note that, because of the special choice (118), in which the initial q-dependence of the fluctuation is the same as the density profile, the destruction fragment can also be written in the (complete and abridged) forms:

$$D(q,K;\tau)=\varDelta(q,K;\tau)\,\varphi_0(q)\equiv\varDelta_\tau\varphi_0(q) \tag{121}$$

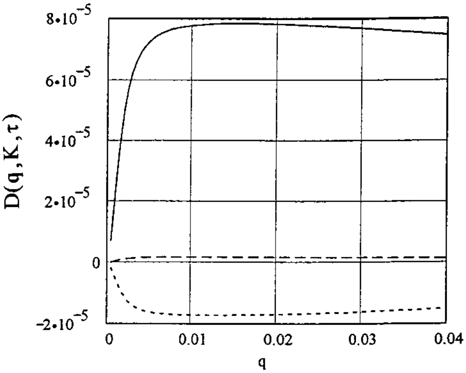

Fig. 15. Contribution of the destruction fragment to the kinetic equation, $D(q, K; \tau)$ at various times. Initial condition: single harmonic $\cos 2\pi\mu\theta$, $\mu = 2$. Solid: $\tau = 1$; dotted: $\tau = 3$; dashed: $\tau = 4$. $K = 22.5$.

The contribution of the destruction fragment to the kinetic equation is shown in Fig. 15. Its absolute value is, indeed very small ($\sim 10^{-5}$), and is very sharply decreasing in time, as expected: it becomes negligible after the short memory time $\tau_M \approx 4$ [see Eq. (46)].

We now consider the solution of the complete master equation (69) for the density profile, including the destruction fragment. It is clearly an inhomogeneous equation associated with the homogeneous equation (87), the destruction fragment playing the role of a source term. We introduce again a propagator $G(q, K; \tau) \equiv G_\tau$, and write the solution in the following form [using the special property (121)]:[12]

$$\varphi(q, K; \tau) = G(q, K; \tau)\, \varphi_0(q) \tag{122}$$

The propagator obeys the following equation:

$$\begin{aligned} G(q, K; \tau + 1) &= \sum_{\sigma=0}^{\tau_C} \psi(q, K; \sigma)\, G(q, K; \tau - \sigma) + \Delta(q, K; \tau + 1) \\ G(q, K; 0) &= 1 \end{aligned} \tag{123}$$

It is not difficult to solve this equation for G_τ in terms of the propagator H_τ of the homogeneous equation, obeying Eq. (89). The first terms are obtained by direct substitution and use of Eq. (90):

---

[12] The forthcoming derivation is easily generalized to the case of an arbitrary q-dependence of the initial fluctuation.

$$
\begin{aligned}
&G_0 = 1\\
&G_1 = H_1 + \varDelta_1\\
&G_2 = H_2 + H_1 \varDelta_1 + \varDelta_2\\
&G_3 = H_3 + H_2 \varDelta_1 + H_1 \varDelta_2 + \varDelta_3\\
&G_4 = H_4 + H_3 \varDelta_1 + H_2 \varDelta_2 + H_1 \varDelta_3 + \varDelta_4
\end{aligned}
\tag{124}
$$

This suggests the form of the general solution for $\tau > 4$:

$$G_\tau = H_\tau + \sum_{\sigma=1}^{4} H_{\tau-\sigma} \varDelta_\sigma, \qquad \tau > 4 \tag{125}$$

This relation is easily proved by induction. Assume its validity for all times less than or equal to τ. The propagator at time $\tau + 1$ obeys Eq. (123); substituting (125) into the right hand side of the latter we obtain:

$$
\begin{aligned}
G_{\tau+1} &= \sum_{\rho=0}^{4} \psi_\rho \left[H_{\tau-\rho} + \sum_{\sigma=1}^{4} H_{\tau-\rho-\sigma} \varDelta_\sigma \right] + \varDelta_{\tau+1}\\
&= \sum_{\rho=0}^{4} \psi_\rho H_{\tau-\rho} + \sum_{\sigma=1}^{4} \sum_{\rho=0}^{4} \psi_\rho H_{\tau-\rho-\sigma} \varDelta_\sigma + \varDelta_{\tau+1}\\
&= H_{\tau+1} + \sum_{\sigma=1}^{4} H_{\tau-\sigma+1} \varDelta_\sigma + 0 \qquad \text{Q.E.D.}
\end{aligned}
$$

In the last step we used twice Eq. (89), and the fact that $\varDelta_\tau = 0$ for $\tau > 4$.

The form of Eq. (125) is not surprising. If both sides are multiplied by the initial condition φ_0, it expresses, in discrete time, a well-known property of linear equations: the solution of the initial value problem for the inhomogeneous equation[13] (69) equals the corresponding solution of the homogeneous equation (87) plus the convolution of the homogeneous propagator with the source term.

Although this property is almost trivial, we wish to stress here one of its consequences, that may help clarifying some ideas. It is usually stated that for times longer than the memory time (or the duration of a collision in kinetic theory), $\tau > \tau_M$, the destruction operator can be set to zero in the master equation (69): this is part of the markovianization process [see Sections 2 and 3]. This statement is certainly correct. It does *not* imply, however, that the *solution* of the asymptotic homogeneous equation (89) equals (approximately) the *solution* of the inhomogeneous equation (69)

---

[13] The sum in the convolution is truncated at $\tau = \tau_C$, as in Eq. (87).

for all times $\tau > \tau_M$. Equation (125) shows that *for all times* $\tau > \tau_M$ the solution of the "exact" equation contains a contribution from the destruction term. This contribution is in the form of a convolution of the destruction fragment with the homogeneous propagator over a time range equal to τ_M. It is a function of time that decays over a time scale determined by the homogeneous propagator. It eventually relaxes to the solution (88), but only after a time much longer than τ_M. In conclusion:

- *The complete (inhomogeneous) master* ***equation*** *(69) reduces to the homogeneous* ***equation*** *(87) after a* ***short time*** $\tau = \tau_M$;
- *The* ***solution*** *of the initial value problem for the inhomogeneous master equation (122) relaxes (diffusively) to the corresponding* ***solution*** *of the homogeneous equation (88) on the* ***long time*** *scale* $\tau = \tau_R$.

This behavior is illustrated in Fig. 16, where the contribution $\delta(q, K; \tau) = \sum_{\sigma=1}^{4} H_{\tau-\sigma} \Delta_\sigma \varphi_0(q)$ of the destruction fragment to $\varphi(q, K; \tau)$ is plotted against q for increasing times. The slow decay of this function is clearly visible. After a short time, the decay is very nearly exponential (Fig. 17) and is fitted by exactly the same diffusive law as the relaxation of the density profile itself [Eqs. (104)–(107)]. This confirms the point stressed above: the relaxation of the destruction fragment contribution is controlled by the homogeneous propagator.

We have also investigated the effect of different initial conditions, in particular Eq. (117) for $\mu = 1$, 3, 4, as well as the same equation with $\sin 2\pi\mu\theta$ replacing $\cos 2\pi\mu\theta$. The qualitative behavior in all these cases is quite similar to the case discussed in detail above and will not be further

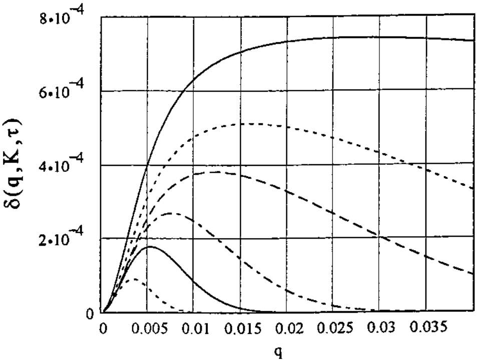

Fig. 16. Contribution of the destruction fragment to the non-Markovian solution for the density profile, $\delta(q, K; \tau)$, vs. q, at various times. $\mu = 2$; $K = 22.5$. From top to bottom: $\tau = 1$, 3, 10, 50, 150, 500.

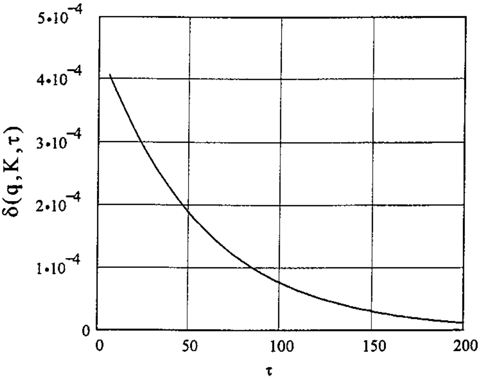

Fig. 17. Contribution of the destruction fragment to the non-Markovian solution for the density profile, $\delta(q, K; \tau)$, vs. τ, at fixed $q = 0.013$. $\mu = 2$; $K = 22.5$.

illustrated here. The destruction fragment contributions in these cases only differ in absolute value (and possibly in sign). It may be remarked that the higher harmonics ($\mu > 2$) of the initial fluctuation give rise to smaller contributions to the density profile.

We finally note that the graphs of $\delta(q, K; \tau)$ as functions of K, for fixed q and τ, exhibit the unavoidable oscillations due to the Bessel functions.

10. THE FLUCTUATIONS

We now go over to the study of the fluctuations, determined by Eqs. (72)–(74). Many features will be familiar by now, hence the discussion may be shorter. The expressions of the basic operators are now more cumbersome. We therefore limit ourselves to their expression through order $(qK)^2$; the rapid convergence (which we checked) justifies this approximation. For convenience, we considered fluctuations containing a single cosine harmonic.[14] As a result, the fluctuations will be grouped as follows:

$$g_{Cm}(q, K; \tau) = \tfrac{1}{2}[g_m(q, K; \tau) + g_{-m}(q, K; \tau)], \qquad m > 0 \qquad (126)$$

with a similar notation for the basic creation and propagation operators.

The explicit expression of the first few creation operators, for the harmonics $m = 1, 2$, is obtained from Eq. (73) for $\tau = 2, 3$:

---

[14] We also considered the sine harmonics, which turn out to have a basically similar behavior. They will not be discussed in this paper.

$$\begin{aligned}
&\mathscr{C}_{C1}(q, K; 2) = \tfrac{1}{2} J_1(qK)[J_2(K - qK) - J_2(K + qK)] \\
&\mathscr{C}_{C1}(q, K; 3) = J_2(qK)[J_0(K)\, J_3(2K) + J_4(K)\, J_5(2K)]
\end{aligned} \tag{127}$$

$$\begin{aligned}
\mathscr{C}_{C2}(q, K; 2) = {}& \tfrac{1}{2} J_2(qK)[J_4(2K - qK) + J_4(2K + qK)] \\
\mathscr{C}_{C2}(q, K; 3) = {}& \tfrac{1}{2} J_1(qK)\{ -[J_0(K + qK)\, J_3(2K + qK) \\
& - J_0(K - qK)\, J_3(2K - qK)] + [J_4(K - qK)\, J_5(2K + qK) \\
& - J_4(K + qK)\, J_5(2K - qK)]\} + J_2(qK)\, J_6^2(2K)
\end{aligned} \tag{128}$$

We do not continue this list explicitly. Let it be mentioned that for $\tau \geqslant 4$ the expressions for the creation operators involve infinite sums that have to be truncated, as explained in Appendix B. The propagation fragment is even more cumbersome. We only give here, for illustration, its expression for the harmonic $m = 1$, assuming the initial condition to be Eq. (118) with $\mu = 2$:

$$\begin{aligned}
\mathscr{P}_{C1,2}(q, K; 3) = {}& \frac{a}{4} J_1(qK)[J_2(K + qK) - J_2(K - qK)]\, \varphi_0(q) \\
\mathscr{P}_{C1,2}(q, K; 4) = {}& \frac{a}{4} \{J_1(qK)[J_5(K + qK)\, J_7(3K - qK) - J_5(K - qK) \\
& \times J_7(3K + qK) + J_1(K + qK)\, J_5(3K - qK) \\
& - J_1(K + qK)\, J_5(3K - qK)] + J_0(qK)[J_0(K + qK) \\
& \times J_3(2K + qK) + J_0(K - qK)\, J_3(2K - qK) \\
& + J_4(K + qK)\, J_5(2K + qK) + J_4(K - qK)\, J_5(2K - qK)] \\
& + 2J_2(qK)[J_6(K)\, J_9(4K) + J_2(K)\, J_7(4K)]\}\, \varphi_0(q)
\end{aligned} \tag{129}$$

In Fig. 18 the creation operator $\mathscr{C}_{C1}(q, K; \tau)$ is plotted against q for $\tau = 2, 3, 4$ ($K = 25$). We see again that it decays very rapidly, on the same short memory time scale τ_M as the basic operators for the density profile. The same rapid decay is found for the propagation operator $\mathscr{P}_{C1,2}(q, K; \tau)$. The dependence on K of the creation operator at various times, exhibiting the ubiquitous oscillations, is shown in Fig. 19.

We now consider the expression of the fluctuations at time τ. We stress again the following important fact. Contrary to the density profile, *there is **no need of solving an equation** for obtaining this quantity. The expression for the density profile—known from the previous step—is simply to be inserted into Eq. (72).* In particular, for $\tau > \tau_M$, the propagation fragment has

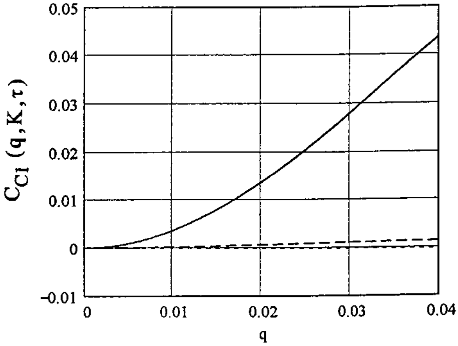

Fig. 18. Creation operator $\mathscr{C}_{C1}(q, K; \tau)$ vs. q. $K=25$. Solid: $\tau=2$; dotted: $\tau=3$; dashed: $\tau=4$.

decayed to practically zero and no longer contributes to the fluctuations. The expression for the latter thus reduces to:

$$g_{Cm}(q, K; \tau)=\mathscr{C}_{Cm}(q, K; 2)\, \varphi(q, K; \tau-2)+\mathscr{C}_{Cm}(q, K; 3)\, \varphi(q, K; \tau-3) + \mathscr{C}_{Cm}(q, K; 4)\, \varphi(q, K; \tau-4) \tag{130}$$

This expression is to be compared to the asymptotic Markovian approximation:

$$g_{M,\, Cm}(q, K; \tau)=[\mathscr{C}_{Cm}(q, K; 2)+\mathscr{C}_{Cm}(q, K; 3)+\mathscr{C}_{Cm}(q, K; 4)]\, \varphi_M(q, K; \tau) \tag{131}$$

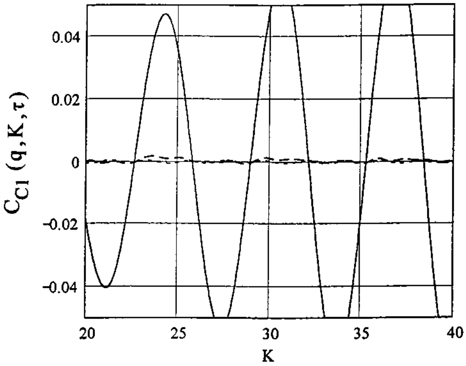

Fig. 19. Creation operator $\mathscr{C}_{C1}(q, K; \tau)$ vs. K. $q=0.035$. Solid: $\tau=2$; dotted: $\tau=3$; dashed: $\tau=4$.

These results show that *the initial fluctuations, propagated by* $\mathscr{P}_{m,m'}(q;\tau)$, *die away very quickly* ($\tau \sim \tau_M$). *They are replaced by fresh fluctuations created out of the density profile: these are long-lived* ($\tau \sim \tau_R$). *They will, however, also decay to zero as the system evolves diffusively toward a spatially homogeneous state.* Here we see a difference with the otherwise analogous evolution of a gas. In the latter case, the correlations created out of the "vacuum" evolve toward a non-zero asymptotic value representing the equilibrium correlations (due to the intermolecular forces).

For the illustrations in the present section we chose a slightly different default value for $K=25$; this is because for $K=22.5$ the creation fragment is close to zero, as can be seen in Fig. 19. In Fig. 20 we plotted the fluctuations

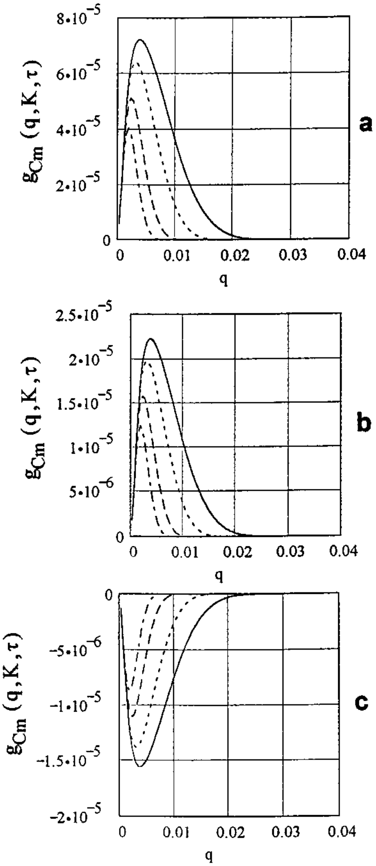

Fig. 20. Fluctuations $g_{Cm}(q, K; \tau)$ vs. q for different harmonics at different times. $K=25$. a: $m=1$; b: $m=2$; c: $m=3$. From top to bottom: $\tau=50$, 100, 250, 500.

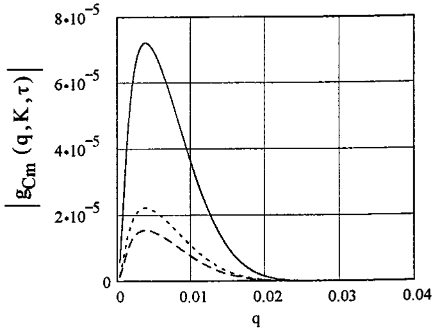

Fig. 21. Fluctuations $g_{Cm}(q, K; \tau)$ vs. q for different harmonics at fixed time $\tau = 50$. $K = 25$. Solid: $m = 1$; dotted: $m = 2$, dash-dotted: $m = 3$.

as functions of q at several times for the three leading harmonics ($m = 1, 2, 3$): their slow decay is clearly visible. The decay is faster, as usual, for the large q-tail. Moreover, the lower harmonics dominate in these "natural" fluctuations, as can be seen in Fig. 21.

An interesting test of the previously described mechanism of evolution is provided by a study of the asymptotic time dependence of the fluctuations. This is shown in Fig. 22 for a fixed value of q. The curves are very nearly exponential for all three harmonics, as can be clearly seen by plotting the natural logarithm of the fluctuations against time (Fig. 23). The straight lines corresponding to the three harmonics are exactly parallel, indicating that the relaxation time associated with all the harmonics is the same. This indicates that for the value of q considered, the fluctuations have very nearly reached the asymptotic Markovian regime (131). In this regime the time dependence is solely contained in the density profile $\varphi(q, K; \tau)$ for all harmonics. As a final test, we compare the numerically

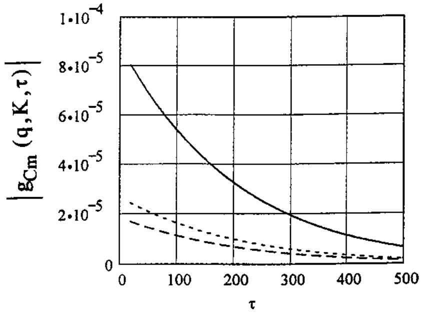

Fig. 22. Asymptotic time dependence of the fluctuations $g_{Cm}(q, K; \tau)$. $q = 0.005$, $K = 25$. Solid: $m = 1$; dotted: $m = 2$, dash-dotted: $m = 3$.

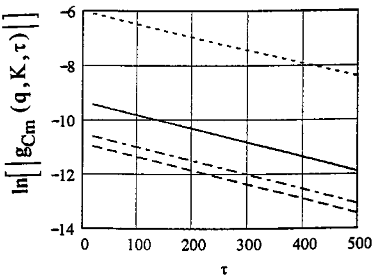

Fig. 23. Natural logarithm of the fluctuations $|g_{Cm}(q, K; \tau)|$. $q = 0.005$, $K = 25$. Solid: $m = 1$; dash-dotted: $m = 2$, dashed: $m = 3$. The upper dotted line has a slope equal to $\tau_R^{-1}(q, K)$.

obtained relaxation time of the fluctuations with the diffusive relaxation time of the density profile [see Eq. (106)]: $\tau_R^{-1}(q, K) = (2\pi q)^2 D(K)$, with the value of the diffusion coefficient for $K = 25$ determined from Eq. (114): $D(25) = 4.841$. The result is represented in Fig. 23 by the dotted straight line with a slope equal to τ_R^{-1}: it is strictly parallel to the three other lines. This is the final check of the mechanism of evolution of the fluctuations described above.

11. DISCUSSION OF EARLIER WORKS

The (vast) literature on the standard map may be divided into two groups containing, respectively, "local" theories and "global" theories. The former analyze individual trajectories, solutions of Eq. (2), with the purpose of establishing their topological properties as revealed typically by time-correlation functions. In the globally stochastic region, such studies provide us with a very detailed picture of the motion of an individual point, with alternations of quasi-random motion and of segments sticking to islands, as well as the effect of this topology on transport. Most of these studies must rely on very long and precise numerical computations of a large number of orbits. (The most recent examples of high-resolution computations, for K near the threshold of stochasticity, are found in refs. 26 and 27.) The most typical and important studies of this type are those of Karney and his coworkers(30, 31) in which the sticking property of islands is studied in detail (see below). A different "local" approach, entirely analytical, is the work of Meiss and Ott(32, 33). It describes transport in a non-diffusive regime in terms of first recurrence times. It is applicable to transport across cantori (whenever these exist). This regime is, however, outside the range envisaged in the present work.

The works in the second group involve the study of the distribution function (or of the density profile) of a statistical ensemble, based on a kinetic equation. This global approach is complementary to the previous one. Both are exact, hence equivalent at the starting point, but the type of approximations adapted to each of them is different. Thus, in the global approach it is difficult to describe the fine structure of orbits in the neighborhood of an island. On the other hand, the local approach does not tell us anything about the shape of the density profile. The present work definitely belongs to the second group.

The possible occurrence of "deterministic diffusion" in dynamically unstable systems was discovered a long time ago. The first work treating this problem explicitly is probably due to Zaslavsky and Chirikov.(7) Using a simple Hamiltonian and applying the methods of Prigogine(19) with an assumption of random phases, they derive a Fokker–Planck equation for the distribution of actions. This work treated a variety of dynamical systems described both by continuous-time Hamiltonian dynamics and by maps.

The earliest works on the standard map recognized the relation between this simple model and a diffusive evolution in the limit of large stochasticity parameter K. These results were reviewed in a celebrated paper by Chirikov.(2) The approach followed there was not based on statistical mechanics, but rather on the study of long chaotic trajectories obtained by iteration of Eq. (2) ("local" approach). The diffusion coefficient is obtained by calculating numerically the quantity:

$$D_C = \lim_{\tau \to \infty} \frac{(x_\tau - x_0)^2}{2\tau} \tag{132}$$

An average is taken over a large number of trajectories (assuming the angles θ are randomly distributed). Chirikov derived analytically the quasilinear formula (103). (Chirikov's methodology based on the study of chaotic orbits was later developed and put on a more and more rigorous mathematical basis.(28, 29)) He also fitted the density profile obtained numerically (either from a large number of orbits with random initial conditions, or from a single long orbit) to a Gaussian packet [the inverse Fourier transform of Eq. (108)]. These features led him to conclude that the behavior of the standard map is diffusive for large K. Moreover, he evaluated numerically the function $D(K)$. The latter yielded a rather fuzzy picture,[15] in which the author detected "some periodicity in the dispersion of the experimental points." He then attempted a qualitative explanation of this periodicity in terms of the effect of the "islands of stability" and accelerator modes.

We believe, however, that there exists at present no really satisfactory proof of the relation between K-periodicity and the presence of islands. The

---

[15] This fuzziness was unavoidable with the computing technology of that time!

accelerator modes do explain the *singularities* that appear in the more recent simulations of the function $D(K)$.(14, 33) The detailed discussion of this problem goes beyond the matter treated here. Let us recall that these modes are stable only in certain windows of K,(3, 30, 33–35) whose width decreases with increasing K. The numerically detectable singularities of the diffusion coefficient occur precisely inside the first few of these windows. Any orbit that is sticking to an accelerator mode island is dragged along by the latter; as a result, the mean square displacement grows faster than τ, and the diffusion coefficient (132) diverges.(30, 31, 33) This is certainly a correct explanation of the *singularities*, but not of the *periodicity*.

An interesting alternative approach to the superdiffusive regime produced by accelerator modes is found in refs. 36 and 37. The "strange transport" process is modeled by a random walk of Lévy type, leading to a mean square displacement of the form $\langle x^2(t) \rangle \sim t^\alpha$, with $\alpha > 1$. The diffusion exponent is determined from a numerical investigation of individual standard map orbits.

In our treatment the accelerator modes are wiped out (the same statement holds for the works in refs. 3, 9, 14); the inclusion of their effect requires an extension of the present theory, in particular, a study of the convergence of the Bessel series defining the evolution operators and the diffusion coefficient; this is beyond our present purpose. As a result there appear no singularities in the existing analytical theories of the diffusion coefficient. The periodicity of $D(K)$ is, however, present even in absence of accelerator modes. Moreover, *all* physical quantities exhibit oscillations in K, as was shown for some examples in our work. Technically, these oscillations are due to the fact that the evolution in the standard map dynamics is closely connected to Bessel functions. We do not see any correlation with the presence of islands. For increasing values of K all islands progressively become exceedingly small. The "regular" oscillations of the physical quantities, on the other hand, are present for arbitrarily large K, with approximately constant period.

A second group of works that are relevant to the present problem is due to Rechester, Rosenbluth and White,(8, 9) to which we may add the paper of Cohen and Rowlands.(10) This work is also described and slightly modified in ref. 3. There is an important difference with the present work: Rechester, Rosenbluth and White study a *noisy standard map*. In other words, they assume the presence of an *extrinsic diffusion*, that will combine with the intrinsic dynamical stochasticity. This modification is mainly done for smoothing the numerical simulations. It is, however, not necessary: their results (in the large-K case) become comparable to ours if the extrinsic diffusion is put to zero (as shown also in ref. 3). Note, however, that in the work of Karney *et al.*(30), which is also devoted to noisy standard maps, the role of noise appears in a more interesting way as a test for the presence of accelerator modes by a special dependence of the diffusion coefficient on σ, the variance of the noise: $D \sim \sigma^{-1}$ (the analysis of this work is, however, of the "local" type).

The methodology of ref. 8 can be summarized as a "mixed" treatment of continuous and discrete time quantities. A distribution function in the complete phase space,[16] $F(x, \theta; \tau)$, is introduced as a solution of an evolution equation (assumed, rather than derived):

$$\frac{\partial F}{\partial t} + x\,\frac{\partial F}{\partial \theta} + \dot{x}\,\frac{\partial F}{\partial x} - \frac{\sigma}{2}\,\frac{\partial^2 F}{\partial \theta^2} = 0 \tag{133}$$

where σ is the extrinsic diffusion coefficient. This is a continuous-time description in terms of a "hybrid" Liouville equation with an additional diffusion term. The discrete time aspect is introduced by modelling the force term by a standard map dynamics, i.e., by introducing kicks at discrete times: $\dot{x} = (K/2\pi)\sin 2\pi\theta \sum_j \delta(t-j)$. The evolution of the distribution function in the interval $j-1+\varepsilon < t < j-\varepsilon$ is given by the propagator of the convection-diffusion equation (without the third term). Between $j-\varepsilon$ and $j+\varepsilon$ the function undergoes a jump: $F(\theta, x; j+\varepsilon) = F(\theta, x+(K/2\pi)\sin 2\pi\theta; j-\varepsilon)$. This procedure is analogous to the usual derivation of a map from a differential equation. The authors are not interested in the explicit study of the distribution function itself (for which they obtain a formal solution), but only in the calculation of the diffusion coefficient defined as:

$$D = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} dx \int_0^1 d\theta (x-x_0)^2\, F(x, \theta; T) \tag{134}$$

Their result showed for the first time analytically the appearance of the oscillations announced by Chirikov. Their quantitative expression will be discussed below [Eq. (138)].

In their following paper(9) the authors introduced an important method of investigation of the standard map kinetic equation, based on the Fourier representation. All subsequent works on the problem, including the present one, used this representation as a starting point. Its application to the solution of the equation of evolution obtained in the previous paper leads to the selection of contributions, as discussed in Appendix B of our present paper. Their criterion for this selection is the large value of K. The smallness of q is not used explicitly; it appears, however, implicitly in the fact (mentioned above) that the authors are interested only in the calculation

---

[16] For the reader's convenience, we use here our own notations, which are rather different from those of Rechester and White.

of the diffusion coefficient. The latter is a quantity defined as a limit for $q \to 0$ [see Eq. (113)]. The authors of ref. 9 devise an ingenious graphical method ("paths") for the characterization of the various contributions. This allows them to derive an expression for the diffusion coefficient involving terms with up to three Bessel functions. It may be noted at present that these authors do not use a closed equation for the density profile. As will be seen below, this leads them to including some Fourier paths that are not allowed in a correct kinetic theory. The authors devote the second part of their paper to the study of the small-K regime. We do not discuss this matter, because we have not considered this regime here.

A continuation of this work is contained in the paper of Meiss *et al.*,(11) which addresses the more general problem of the correlation functions of two-dimensional, periodic, area-preserving maps, with the standard map as a particular case. The expression of the diffusion coefficient appears as a by-product of this theory. In this work use is made of characteristic functions defined as averages over initial conditions of functions of the orbits, rather than of distribution functions in the sense of statistical mechanics. A kind of renormalization procedure is introduced, consisting in the summation of a partial infinite series of selected terms contributing to the diffusion coefficient. It is, however, clear that the exclusions due to the projectors P and Q that appear in kinetic theory are not taken into account either here.

A different approach appears in the book of Lichtenberg and Lieberman;(3) it was first introduced by the same authors(12) for a different map. This treatment does not introduce any extrinsic diffusion, in contrast to the previous work. Instead, these authors introduce from the very beginning a Chapman–Kolmogorov equation, from which they derive a Fokker–Planck equation without more than a qualitative justification. In a sense, the stochastic diffusion process is *assumed a priori* rather than derived from the dynamics. The transition probabilities are determined from the standard map equations. From the fundamental point of view this procedure is rather unorthodox, by mixing concepts from Hamiltonian dynamical systems and probability theories (Markov processes). This mixture is avoided in our present treatment, which starts from the purely dynamical Perron–Frobenius operator. Moreover, the Lichtenberg–Lieberman treatment is based from the beginning on a continuous-time formalism. Their subsequent calculations are very similar (Fourier path method) to those of Rechester, Rosenbluth and White.[17]

---

[17] Note that in refs. 3, 8, 9, 11 a version of the standard map slightly different from Eq. (2) is used: $x_{\tau+1} = x_\tau + K \sin \theta_\tau$, $\theta_{tau+1} = \theta_\tau + x_{\tau+1}$ (mod 2π). This leads to some unimportant differences (due to factors 2π) in the final results.

A similar work is due to Abarbanel and Crawford.$^{(13), 18}$ They calculate a conditional probability $P(\mathbf{z}, t \mid \mathbf{w}, t_0)$ that the system is at $\mathbf{z}$ at time t, given that it was at $\mathbf{w}$ at time t_0. The time is then discretized and from the former object a diffusion coefficient is calculated. The formalism is applied to a variant of the standard map, and use is made of the Fourier paths described above.

A very interesting work in this domain is due to Abarbanel.$^{(38)}$ It is devoted to the "non-linear diffusion in Hamiltonian systems exhibiting chaotic motion". Its distinguishing feature is the derivation of a true master equation for the reduced distribution of actions, by using Zwanzig's projection technique. In this sense, this work is close to ours. He uses, however, a continuous-time formalism; hence his master equation is of the type of Eq. (9) rather than of Eq. (41). His equation is applied, in particular, to the standard map, but not for the same purposes as in the present paper. He is mostly interested in calculating an autocorrelation function and showing that it decays exponentially in time, a characteristic of mixing systems. From our present point of view, the most interesting feature of Abarbanel's work is the stress put on the difference between "direct dynamics," governed by the Liouvillian $\mathscr{L}$, and "projected dynamics", determined by the operator $Q\mathscr{L}$. Abarbanel's master equation (Eq. (20) of his paper) is:

$$\frac{\partial}{\partial t} F(\mathbf{I}; t) = \int_0^t d\tau \, \frac{\partial}{\partial I_j} D_{jk}(\mathbf{I}, \tau) \frac{\partial}{\partial I_k} F(\mathbf{I}, t-\tau) + (\text{destruct}) \tag{135}$$

where $F(\mathbf{I}, t)$ is the reduced distribution function of the actions I_j, and $D_{jk}(\mathbf{I}, \tau)$ is the "projected diffusion tensor":[19]

$$D_{jk}(\mathbf{I}, \tau) = P \frac{\partial H_1}{\partial \theta_j} \exp(-Q\mathscr{L}\tau) \frac{\partial H_1}{\partial \theta_k} \tag{136}$$

with H_1 the perturbation Hamiltonian and P, Q the projection operators defined in Section 3 (we do not write down here the expression of the destruction term). Clearly, it is the projected Liouvillian $Q\mathscr{L}$ that determines the evolution in the intermediate states, not the full Hamiltonian, which defines a "direct diffusion tensor" in Abarbanel's terminology:

$$\varDelta_{jk}(\mathbf{I}, \tau) = P \frac{\partial H_1}{\partial \theta_j} \exp(-\mathscr{L}\tau) \frac{\partial H_1}{\partial \theta_k} \tag{137}$$

He derived an elegant operator equation relating the two diffusion tensors. It appears that the works reviewed above$^{(3, 8, 9, 11)}$ use the direct dynamics

---

[18] It is worth noting that this paper is misreferred to by Lichtenberg and Lieberman$^{(3)}$ as (our) ref. 31.

[19] Not to be confused with the diffusion coefficient appearing in the present work!

[corresponding to Eq. (137)] rather than the correct projected dynamics [corresponding to Eq. (136)]. This explains the discrepancy between their results and those of the present work for the diffusion coefficient, as will now be shown.

We first compare the leading terms in the diffusion coefficient, divided by the quasilinear coefficient: $d(K)=D(K)/D_{QL}(K)$ in the various works, which are abbreviated as follows: *RW*: ref. 8; *RRW*: ref. 9; *M*: ref. 11; *LL*: ref. 3; *B*: present work (we use our notations):

$$d_{RW}(K)=1-2J_2(K)\,e^{-\sigma}-2J_1^2(K)\,e^{-\sigma}+2J_3^2(K)\,e^{-3\sigma} \tag{138}$$

$$d_{RRW}(K)=1-2J_2(K)\,e^{-\sigma}-2J_1^2(K)\,e^{-\sigma}+2J_2^2(K)\,e^{-2\sigma}+2J_3^2(K)\,e^{-3\sigma} \tag{139}$$

$$d_{LL}(K)=1-2J_2(K)-2J_1^2(K)+2J_2^2(K)+2J_3^2(K) \tag{140}$$

$$d_{B}(K)=1-2J_2(K)+2J_2^2(K)+2J_3^2(K) \tag{141}$$

The exponential factors in Eqs. (138) and (139) are clearly due to the inclusion of extrinsic diffusion in the RW and RRW models. When σ is set to zero, the expression (139) reduces to the LL result (140) (this is no surprise!). Interestingly, the term $2J_2^2(K)$ was missed in the first RW paper. This is because the term $2J_3^2(K)$ originates from $\psi(q, K; 3)$ [first term in the right hand side of Eq. (85)], whereas the term $2J_2^2(K)$ comes from $\psi(q, K; 4)$, which was not retained in RW [in their terminology, the former comes from a 4-step path, where as the latter comes from a 5-step path]. The most interesting difference between our result and Eq. (140) is the absence in $d_B(K)$ of the term $2J_1^2(K)$. This term in all three previous equations results from the violation of the rules of "projected dynamics". We showed that the memory kernel must be constructed by using only intermediate states with indices $m_j\neq 0$ in Eq. (68). It appears clearly from the RRW path corresponding to this contribution (Fig. 24) that the path goes through $m=0$ in the intermediate state. For the same reason, the higher order corrections of RRW are not comparable to ours. It is, however, remarkable that Eqs. (140) and (141) yield in practice very similar results (Fig. 25).

The works reviewed above are based either on a continuous-time formalism or on a "hybrid" formalism (RW, RRW). We now come to the discussion of works using a formalism explicitly designed for discrete-time maps. There is a large number of such studies; we shall only discuss those that are rather closely related to the spirit of the present paper. The main purpose of these studies is to understand the emergence of an irreversible (possibly diffusive) behavior in conservative dynamical systems. This has been a long-standing objective of Prigogine and his coworkers in Brussels

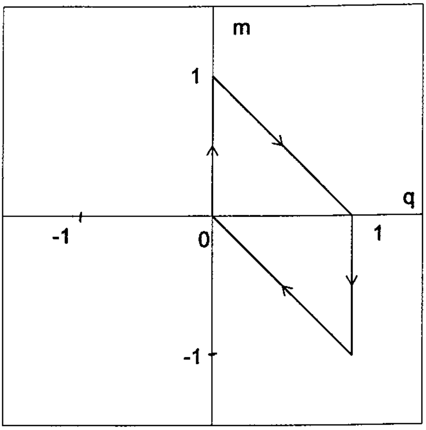

Fig. 24. A Fourier path in the Rechester–Rosenbluth–White theory which violates the rules of "projected dynamics" (it passes through $m = 0$ in an intermediate state).

and in Austin.(3, 16, 18–20, 40, 41) The most recent version of this philosophy is the search for a spectral decomposition of the resolvent of the Liouville equation in an extended functional space, in order to isolate poles corresponding to exponentially decaying states (ref. 42 and references therein). These works treated either simple mechanical models (such as the Friedrichs model) or "large Poincaré systems" (including classical and quantum gases of interacting particles), in a continuous-time formalism.

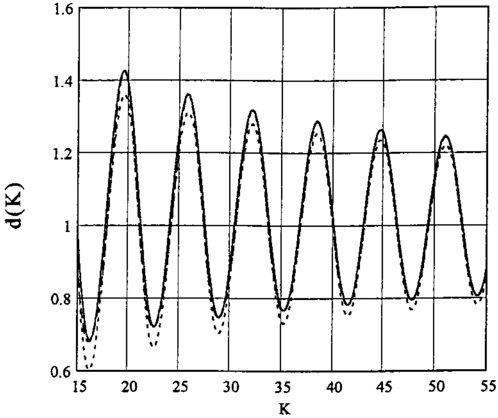

Fig. 25. Comparison of three results for the diffusion coefficient (normalized to the quasilinear value) $d(K)$. Solid: Eq. (141), dotted: Eq. (138) with $\sigma = 0$, dashed: Eq. (140).

The same purpose was also pursued in the study of dynamical systems defined in discrete time. The models treated exhaustively along these lines were simple maps, such as the tent map, the Bernoulli map, the baker and multibaker maps.(39, 43, 44) A review of these methods is given in refs. 21 and 45, in which the discrete-time formalism used in these works is clearly exposed. The projection operator method is used extensively, but the definition of the latter is different from the one used here. A projection operator is attached to each pole of the Perron–Frobenius operator and defines a "subdynamics" in the sense of ref. 40. The eigenvalues obtained in this way are associated with exponentially decaying modes (the corresponding right and left eigenvectors are very complicated mathematical objects). They are related to the so-called "Ruelle resonances".(46) (A related, but somewhat different approach is found in the works of Gaspard, collected in a recent book.(47)) The diffusion coefficient is deduced from the eigenvalue corresponding to the slowest decay rate. Its value obtained in this way for these simple system is essentially the quasilinear value.

These methods were applied to the standard map by Hasegawa and Saphir(14) (see also ref. 6). This work contains a rigorous derivation of the eigenvalues of the Perron–Frobenius operator for the standard map, from which an approximate expression for the diffusion coefficient is obtained: it corresponds to the first two terms in the right hand side of Eqs. (138)–(141):

$$d_{HS}(K) = 1 - 2J_2(K) \tag{142}$$

Unfortunately, a paper containing the details of the calculations, announced by the authors in ref. 14 was never published!

The works discussed above display an interesting mathematical structure of the evolution operator. The spectral decomposition is not, however, of much help in obtaining a picture of the actual shape of the distribution function or of the density profile of these systems. Indeed, the (Gelfand–Schwartz) distributions appearing as eigenstates have no observable physical meaning *per se*. The observable density profiles must be constructed as superpositions of an infinite number of such singular objects.[20] The direct study of physically observable objects was the main motivation for our own present work. In order to achieve this purpose, we do not study the separate eigenvalues and eigenstates, thus partitioning the functional space into an infinite number of subspaces, each one attached to a specific eigenstate. We rather adopt a much simpler partition into two subspaces: one containing the angle-averaged distribution function (the density profile),

---

[20] The situation is similar to the van Kampen–Case modes, well known as singular eigenfunctions of the Vlasov equation of plasma physics,(48, 49) from which the physical distribution functions are constructed by weighted integration (see also ref. 20).

and the other all the rest (the fluctuations). This philosophy closely resembles the one used in our approach to statistical mechanics, as sketched in Section 2, and in more detail in ref. 6 (see also ref. 18). By this procedure the physically interesting part is identified and an appropriate mathematical tool is constructed for its direct and complete study.

An early work in this direction was done by Petrosky.(50) He considered an elastic pendulum (i.e., a weight attached to a spring) in its motion in the chaotic layer near the separatrix in phase space. The problem is treated in continuous time, starting from a two-degrees of freedom Hamiltonian, hence from a Liouville equation for the distribution function. The coupling between the spring and the weight is supposed to be small. The relevant part of the distribution function (of pendulum and string variables) is the reduced distribution function of the two actions. Using a simple version of the non-equilibrium statistical mechanics formalism (Section 2), i.e., second-order perturbation theory, he derives an (asymptotic) Markovian kinetic equation for the reduced distribution function. It has the form of a Fokker–Planck equation, hence it describes a kind of diffusive motion in actions space. Its properties were, however, not studied in detail.

The formalism of non-equilibrium statistical mechanics and of kinetic theory, extensively developed for many-particle interacting systems, remained to be adapted to simpler chaotic dynamical systems described by discrete-time iterative maps. The first important step in this direction was taken by the work of Bandtlow and Coveney,(15) who derived the master equation for such systems. This work was reviewed and completed in Section 3. Their paper was, however, mostly devoted to the mathematical aspects of the problem, such as the precise criteria of applicability of the theory. In several other works, Coveney and his coworkers applied the formalism to various maps (essentially variants of the multi-baker map) in order to study the asymptotic decay of correlation functions of deterministic, reversible, chaotic dynamical systems.(51–53) These studies are also of essentially mathematical nature. We therefore considered that a more physical (and graphical) approach of the problem was desirable: this idea led to the present work.

12. CONCLUSIONS

"Deterministic diffusion" or "Hamiltonian stochasticity" is a concept which has been identified since more than 25 years as the *seemingly* random behavior of certain dynamical systems in certain ranges of the relevant parameters. Many different approaches have been used for its description, each one uncovering some aspects of the phenomenon. Nevertheless, some important facts do not seem to have been studied *systematically* in the past. Diffusion is a paradigm of a transport process. Transport theory is an important part of nonequilibrium statistical mechanics, in particular of kinetic theory. The latter is based on the study of a *kinetic equation* (such as, typically, the Boltzmann equation for dilute gases), i.e., a closed equation of evolution for a "relevant" part of the phase space distribution defining the statistical state of the system. The irreversible behavior emerges as a long-time, asymptotic property of the solutions of the kinetic equation for a certain class of dynamical systems. Traditionally, the systems considered were large many-particle systems of interacting particles. Since the "chaos revolution" the possibility of a kinetic description of small chaotic systems appeared possible and necessary. The chronology of the previous works leading to the present work was discussed in Section 11. We now summarize our main results.

The class of dynamical systems considered here are governed by two-dimensional area-preserving iterative maps. The variables chosen for their description are "action-angle-like." Rather than studying single trajectories, we consider here statistical ensembles of systems whose state is defined by a phase-space distribution function. The latter is uniquely decomposed into an angle-average part (the *density profile*) and a complementary part (the *fluctuations*) by means of an appropriate pair of projection operators. As shown in Section 3, an application of the methods of nonequilibrium statistical mechanics leads to the derivation of a pair of exact equations: a closed master equation (41) for the density profile, and an equation (42) expressing the fluctuations as a functional of the density profile. Both equations apply specifically to the discrete-time nature of the systems considered. All the evolution operators appearing in these equations are derived *exactly* from the basic Perron–Frobenius operator.

The general formalism is applied to the Chirikov–Taylor standard map, for which exact, explicit equations of evolution (58), (61) are obtained, that are valid for any value of the stochasticity parameter. The main characteristic of the master equation (58) is its non-Markovian character in time; in addition, it is also non-local in the wave vector. In other words, the value of the density profile (in the Fourier representation) $\varphi(q, K; \tau)$ depends on its values at all previous times and all wave vectors: $\varphi(q', K; \tau - \sigma)$. An exact general solution of such an equation is out of question, as should be expected. There exists, however, one regime (65) in which the kinetic equations are much simpler (they are no longer non-local in q), Eqs. (69), (72): it corresponds to small values of the wave vector and large values of the stochasticity parameter, coupled according to Eqs. (64) and (65). Under these conditions, the equations become tractable, and approximate solutions can be found analytically. Moreover, it appears that in the long-time limit the master equation can be markovianized, becoming local in time as well. The subsequent work is devoted to finding the conditions under which this simplification occurs and describing in detail the way in which these simpler descriptions emerge asymptotically.

Two characteristic time scales were identified. The short *memory time* τ_M measures the range of the memory in the evolution operators: it is of the order of 4–5 iterations. The long *relaxation time* $\tau_R(q, K)$ essentially measures the rate of diffusive dispersion; it is a decreasing function of the wave vector ($\tau_R \propto q^{-2}$). It is determined by a single diffusion coefficient $D(K)$; the latter oscillates as a function of the stochasticity parameter around the quasilinear value $D_{QL} = K^2/16\pi^2$.

Several distinct relaxation processes can be identified in standard map kinetics.

(a) The density profile evolves from its initial value $\varphi(q, K; 0)$ towards a (time-dependent) function determined by the *Markovian* form of the equation of evolution $\varphi_M(q, K; \tau)$ (briefly called the Markovian solution), Eq. (98);

(b) The Markovian solution evolves towards a Gaussian packet characteristic of a truly diffusive evolution (i.e., a solution of the diffusion equation) $\varphi_G(q, K; \tau)$, Eq. (108);

(c) The Gaussian packet evolves toward a function sharply peaked around $q = 0$ (whose maximum remains, however, equal to one because of the conservation of normalization): it describes the spatial homogenization produced by diffusion;

(d) The initial fluctuations produce during a short time ($\sim \tau_M$) a small cumulative effect in the density profile through the destruction fragment; this contribution lives for a long time ($\sim \tau_R$).

(e) The initial fluctuations die out very quickly ($\sim \tau_M$) and are replaced by long-living fluctuations created out of the density profile; they eventually ($\sim \tau_R$) decay to zero.

All these processes are characterized by an initial transient regime, followed by an *exponential decay* of the form (104). The remarkable fact is that *the relaxation time associated with all these processes is the same*: $\tau_R^{-1}(q) = (2\pi q)^2 D$. This does not imply, however, that these processes are simultaneous; their transient stages may, indeed, be different. We have shown the "exact" (non-Markovian) solution running after the Markovian solution, which is itself running after the Gaussian; in spite of the equality of relaxation times, they do catch up one after the other (Fig. 14). The evolution of the density profile described above is easily understood from the (diffusive) form of the relaxation time. The decay of the large-q region is faster than the core. The tail of the of the distribution function is thus progressively shortened, and the density profile narrows toward $q = 0$.

The Master equation provides us with a very detailed picture of the evolution of a system governed by the standard map. This approach goes much further than the mere determination of a diffusion coefficient. It allows us to follow in detail the evolution in time of the shape of the density profile and of the fluctuations.

Moreover, the results obtained here can be extrapolated, at least qualitatively, to the behavior of more complicated systems, including the many-particle systems of interest in statistical mechanics. In the sense, the standard map can be considered as a testing bench for kinetic theory.

APPENDIX A. BASIC OPERATORS FOR THE STANDARD MAP

The explicit evaluation of the basic operators entering the master equations for the density profile and for the fluctuations in the case of the standard map is straightforward. They are defined by their matrix elements between states $|q, 0\rangle$ for density profile (P-)states, or $|q, m\rangle$ ($m \neq 0$) for fluctuation (Q-)states. All fragments in Eqs. (34)–(37) are defined in terms of the propagation fragment $Q\mathscr{P}(\tau)\, Q$. The latter is calculated by using the definitions (34) and (56).[21] We find, for $\tau \geqslant 2$:

$$\begin{aligned}
&\langle q, m_0|\, \tilde{\mathscr{P}}(\tau+1)\, |q_\tau, m'\rangle \\
&\quad = \langle q, m_0|\, (Q\mathscr{U}Q)^\tau\, |q', m'\rangle = \sum_{m_1\neq 0} \sum_{m_2\neq 0} \cdots \sum_{m_{\tau-1}\neq 0} \int dq_1\, dq_2 \cdots dq_{\tau-1} \\
&\qquad \times \delta(q_1 - q - m_0)\, J_{m_0-m_1}[(q+m_0)\, K] \\
&\qquad \times \delta(q_2 - q_1 - m_1)\, J_{m_1-m_2}[(q_1+m_1)\, K] \\
&\qquad \vdots \\
&\qquad \times \delta(q_\tau - q_{\tau-1} - m_{\tau-1})\, J_{m_{\tau-1}-m'}[(q_{\tau-1}+m_{\tau-1})\, K] \qquad \text{(A1)}
\end{aligned}$$

Performing the integrations over $dq_1 \cdots dq_{\tau-1}$ we find [according to our convention defined in Section 3, *in all summations over* (*lower case*) m_j *the value* $m_j = 0$ *is excluded*: this is the effect of the projector Q]:

---

[21] Note that, with the convention on the notation of integers defined in Section 3, we have the following relations: $\langle q, M|\, Q\mathscr{U}Q\, |q', M'\rangle = \langle q, m|\, \mathscr{U}\, |q', m'\rangle$, $\langle q, M|\, Q\mathscr{U}P\, |q', M'\rangle = \langle q, m|\, \mathscr{U}\, |q', 0\rangle$, etc.

$$\langle q, m_0|\, \tilde{\mathscr{P}}(\tau+1)\, |q_\tau, m'\rangle$$
$$= \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{\tau-1}} \delta\left(q_\tau - q - m_0 - \sum_{j=1}^{\tau-1} m_j \right) J_{m_0-m_1}[(q+m_0)\, K]$$
$$\times J_{m_1-m_2}[(q+m_0+m_1)\, K] \cdots J_{m_{\tau-1}-m'}\left[\left(q+m_0+\sum_{j=1}^{\tau-1} m_j \right) K \right] \tag{A2}$$

We recognize here the operator $\hat{\mathscr{W}}$ defined in Eq. (57) of the main text, and the relation:

$$\langle q, m_0|\, \hat{\mathscr{P}}(\tau+1)\, |q_\tau, m'\rangle = \hat{\mathscr{W}}_{m_0, m'}(q \mid q_\tau; \tau-1), \qquad \tau \geqslant 2 \tag{A3}$$

For $\tau = 0$ we find, using Eq. (39):

$$\langle q, m|\, \hat{\mathscr{P}}(1)\, |q', m'\rangle = \delta(q'-q)\, \delta_{m, m'} \tag{A4}$$

and for $\tau = 1$ we find simply the matrix element of $Q\mathscr{U}Q$:

$$\langle q, m|\, \hat{\mathscr{P}}(2)\, |q', m'\rangle = \delta(q'-q-m)\, J_{m-m'}[(q+m)\, K] \tag{A5}$$

The remaining operators, $\mathscr{D}$, $\mathscr{C}$, $\mathscr{E}$ are obtained in a quite similar way, by multiplying the matrix $\mathscr{P}$ to the left or/and to the right with a matrix $P\mathscr{U}Q$ or/and $Q\mathscr{U}P$.

APPENDIX B. EVALUATION OF THE MEMORY KERNEL

The approximate evaluation of the memory kernel $\psi(q, K; \tau)$ will be sketched here for $\tau = 0$, 2, 3, 4. For $\tau = 0$, we found already the result in Eq. (70): $\psi(q, K; 0) = J_0(qK)$. This result is complete and exact; the contribution is of order $(qK)^0$. We also found $\psi(q, K; 1) = 0$, a typical result of the diffusive regime. The first non-trivial contribution is found from Eq. (68):

$$\psi(q, K; 2) = \sum_{m_1 \neq 0} \sum_{m_2 \neq 0} \delta_{Kr}(m_1+m_2)\, J_{-m_1}(qK)\, J_{m_1-m_2}[(q+m_1)\, K]\, J_{m_2}(qK)$$
$$= \sum_{m_1 \neq 0} J_{-m_1}(qK)\, J_{2m_1}[(q+m_1)\, K]\, J_{-m_1}(qK) \tag{B1}$$

The diffusive regime constraint $m_1 + m_2 = 0$ has suppressed one infinite summation. We now decide to retain only contributions through order

$(qK)^4$, recalling Eq. (80) this leaves only four terms in the remaining sum: $m_1 = \pm 1, \ \pm 2$. Hence:

$$\begin{aligned}\psi(q, K; 2) = {} & J_1^2(qK)\, J_2(-qK+K) + J_{-1}^2(qK)\, J_{-2}(-qK-K) \\ & + J_2^2(qK)\, J_4(-qK+2K) + J_{-2}^2(qK)\, J_{-4}(-qK-K)\end{aligned}$$

Using now the well-known properties: $J_{-m}(x) = (-)^m J_m(x)$ and $J_m(-x) = (-)^m J_m(x)$, we obtain:

$$\begin{aligned}\psi(q, K; 2) = {} & J_1^2(qK)[J_2(K-qK) + J_2(K+qK)] \\ & + J_2^2(qK)[J_4(2K-qK) + J_4(2K+qK)]\end{aligned}$$

Finally, we note that each bracketted term contains contributions of order 0, 1,... in (qK); hence, through fourth order, we are left with:

$$\psi(q, K; 2) = J_1^2(qK)[J_2(K-qK) + J_2(K+qK)] + J_2^2(qK)\, 2J_4(2K) \tag{B2}$$

Going over to $\tau = 3$ we find, after implementing the constraints $m_2 = -m_1 - m_3$ and $m_2 \neq 0$:

$$\begin{aligned}\psi(q, K; 3) = \sum_{\substack{m_1 \neq 0 \\ (m_1 + m_3 \neq 0)}} \sum_{m_3 \neq 0} & J_{-m_1}(qK)\, J_{2m_1+m_3}[(q+m_1)\, K] \\ & \times J_{-m_1-2m_3}[(q-m_3)\, K]\, J_{m_3}(qK)\end{aligned} \tag{B3}$$

There are now many more terms (but still a finite number!) contributing to order $(qK)^4$: they are identified by the criterion: $|m_1| + |m_3| \leqslant 4$. We thus retain the following couples: $(m_1, m_3) = (1, 3)$, $(1, 2)$, $(1, 1)$, $(1, -2)$, $(1, -3)$, $(2, 2)$, $(2, 1)$, $(2, -1)$, $(3, 1)$, $(3, -1)$, together with the couples with opposite sign: $(-1, -3)$, $(-1, -2)$,..., $(-3, 1)$. [Note that the couples $(1, -1)$, $(2, -2)$ are excluded by the condition $m_2 = -m_1 - m_3 \neq 0$.] Collecting and grouping the terms according to their order in (qK) as was done in Eq. (B2), one obtains the expression (85) of the main text.

We now consider $\tau = 4$, for which the constraint is: $m_2 = -m_1 - m_3 - m_4$:

$$\begin{aligned}\psi(q, K; 4) = \sum_{\substack{m_1 \neq 0 \\ (m_1 + m_3 + m_4 \neq 0)}} \sum_{m_3 \neq 0} \sum_{m_4 \neq 0} & J_{-m1}(qK)\, J_{2m_1+m_3+m_4}[(q+m_1)\, K] \\ & \times J_{-m_1-2m_3-m_4}[(q-m_3-m_4)\, K] \\ & \times J_{m_3-m_4}[(q-m_4)\, K]\, J_{m_4}(qK)\end{aligned} \tag{B4}$$

Here we encounter for the first time a difficulty. We determine the terms of order up to $(qK)^4$ as before by choosing the couples (m_1, m_4) as described above. But now to each of these couples corresponds an *infinite number of terms* with $-\infty < m_3 < \infty$. In order to find a criterion for truncating these infinite subseries we look at the three middle Bessel functions. For each choice of the couple (m_1, m_4) the arguments of the second and fourth of these (neglecting qK) is $(m_1 K)$ and $(-m_4 K)$ which are fixed for each couple; but the argument of the middle one is $(-m_3 K - m_4 K)$. The value of the Bessel functions decreases asymptotically like $[|m_3 + m_4|\ K]^{-1/2}$ (whatever their order!), thus for given K and m_4 they decay like $|m_3 + m_4|^{-1/2}$. We therefore set a limit for the truncation by requiring that *only terms containing Bessel functions with arguments corresponding to* $|m_3 + m_4| \leqslant m_C$ *are retained*, where m_C is some fixed integer. In practice, for the calculation of $\psi(q, K; 4)$ we chose $m_C = 4$.

As an illustration of this procedure, we write down the truncated subseries for the couple $m_1 = 1$, $m_4 = 3$ [i.e., a contribution of order $(qK)^4$]. The only values to be retained for the subseries are then $-7 \leqslant m_3 \leqslant 1$; among these, the values $m_3 = 0$ and $m_3 = -4$ must be excluded (the latter corresponds to $m_2 = 0$). We then find as a factor of $J_1(qK)\, J_3(qK)$ the following sum:

$$\begin{aligned}
&J_{-2}(K+)\, J_{10}(4K+)\, J_{-10}(-3K+) + J_{-1}(K+)\, J_8(3K+)\, J_{-9}(-3K+) \\
&\quad + J_0(K+)\, J_6(2K+)\, J_{-8}(-3K+) + J_2(K+)\, J_2(qK)\, J_{-6}(-3K+) \\
&\quad + J_3(K+)\, J_0(-K+)\, J_{-5}(-3K+) \\
&\quad + J_4(K+)\, J_{-2}(-2K+)\, J_{-4}(-3K+) \\
&\quad + J_6(K+)\, J_{-6}(-4K+)\, J_{-2}(-3K+)
\end{aligned}$$

[we used here the abbreviation $J_m(nK+) \equiv J_m(nK + qK)$]. Clearly, the maximum absolute value of the argument of the middle Bessel function is $4K$. We note that the fourth term contains the factor $J_2(qK)$; it therefore yields (in combination with $J_1 J_3$) a contribution of order $(qK)^6$ and must therefore be neglected (see footnote in Section 6). This analysis is done for each couple (m_1, m_4). The result is a large number (156) of terms which will, of course, not be written down here. They have been retained, however, in the numerical calculation of $\psi(q, K; 4)$. The rate of convergence with increasing m_C is, unfortunately, very slow. Moreover, the calculation of $\psi(q, K; \tau)$ becomes prohibitive for $\tau > 4$. These quantities are, however, very small compared to $\psi(q, K; \tau)$ for $\tau \leqslant 4$.

The calculation of the other basic operators (destruction, creation and propagation operators) are done in the same way as above, by using the same criteria for truncating the series.

APPENDIX C. SOLUTION OF THE NON-MARKOVIAN MASTER EQUATION

Equation (89) is written out explicitly as follows (recalling that $\psi_1 = 0$):

$$H_\tau = \psi_0 H_{\tau-1} + \psi_2 H_{\tau-3} + \psi_3 H_{\tau-4} + \psi_4 H_{\tau-5}, \qquad \tau \geqslant 5 \tag{C1}$$

For times shorter than 5, one easily derives the first lines of Eq. (90) by direct substitution in the appropriate reduced equation. For longer times the solution is written in the form:

$$H_\tau = \psi_0^\tau + a_\tau \psi_0^{\tau-3}\psi_2 + b_\tau \psi_0^{\tau-4}\psi_3 + c_\tau \psi_0^{\tau-5}\psi_4 + d_\tau \psi_0^{\tau-6}\psi_2^2 \tag{C2}$$

As noted in the main text, ψ_0 is of zeroth order, and ψ_σ is neglected for $\sigma > 4$, as well as positive powers of ψ, except ψ_2^2. Substituting Eq. (C2) into (C1), we find:

$$\begin{aligned} H_\tau = \psi_0[&\psi_0^{\tau-1} + a_{\tau-1}\psi_0^{\tau-4}\psi_2 + b_{\tau-1}\psi_0^{\tau-5}\psi_3 \\ &+ c_{\tau-1}\psi_0^{\tau-6}\psi_4 + d_{\tau-1}\psi_0^{\tau-7}\psi_2^2] \\ &+ \psi_2[\psi_0^{\tau-3} + a_{\tau-3}\psi_0^{\tau-6}\psi_2 + \cdots] \\ &+ \psi_3[\psi_0^{\tau-4} + \cdots] + \psi_4[\phi_0^{\tau-5} + \cdots] \end{aligned}$$

Comparing this expression with Eq. (C2), we find the following relations, together with the corresponding "initial conditions":

$$\begin{aligned} &a_\tau = a_{\tau-1} + 1, && a_0 = a_1 = a_2 = 0 \\ &b_\tau = b_{\tau-1} + 1, && b_0 = b_1 = b_2 = b_3 = 0 \\ &c_\tau = c_{\tau-1} + 1, && c_0 = c_1 = c_2 = c_3 = c_4 = 0 \\ &d_\tau = d_{\tau-1} + a_{\tau-3}, && d_0 = d_1 = d_2 = d_3 = d_4 = d_5 = 0 \end{aligned} \tag{C3}$$

These recurrence relations are easily solved as follows:

$$\begin{aligned} a_\tau &= \tau - 2, && \tau \geqslant 2 \\ b_\tau &= \tau - 3, && \tau \geqslant 3 \\ c_\tau &= \tau - 4, && \tau \geqslant 4 \\ d_\tau &= d_{\tau-1} + \tau - 5, && \tau \geqslant 5 \end{aligned} \tag{C4}$$

The last recurrence relation can be easily solved:

$$d_\tau = \tfrac{1}{2}(\tau - 5)(\tau - 6) \tag{C5}$$

REFERENCES

1. R. Balescu, *Phys. Rev. E* **58**:951 (1998).
2. B. V. Chirikov, *Phys. Reports* **52**:265 (1979).
3. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
4. L. E. Reichl, *The Transition to Chaos* (Springer, New York, 1992).
5. A. Lasota and M. C. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, 1985).
6. R. Balescu, *Statistical Dynamics: Matter out of Equilibrium* (Imperial College Press, London, 1997).
7. G. M. Zaslavsky and B. V. Chirikov, *Usp. Fiz. Nauk* **105**:3 (1971).
8. A. B. Rechester and R. B. White, *Phys. Rev. Lett.* **44**:1586 (1980).
9. A. B. Rechester, M. N. Rosenbluth, and R. W. White, *Phys. Rev. A* **23**:2994 (1981).
10. R. H. Cohen and G. Rowlands, *Phys. Fluids* **24**:2295 (1981).
11. J. D. Meiss, J. R. Cary, C. Grebogi, J. D. Crawford, A. N. Kaufman, and H. D. J. Abarbanel, *Physica D* **6**:375 (1983).
12. M. A. Lieberman and A. J. Lichtenberg, *Plasma Phys.* **15**:125 (1973).
13. H. D. J. Abarbanel and J. D. Crawford, *Phys. Lett. A* **82**:378 (1981).
14. H. H. Hasegawa and W. C. Saphir, in: *Aspects of Nonlinear Dynamics*, I. Antoniou and F. Lambert, eds. (Springer, Berlin, 1991).
15. O. F. Bandtlow and P. V. Coveney, *J. Phys. A: Math. Gen.* **27**:7939 (1994).
16. I. Prigogine and P. Résibois, *Physica* **27**:629 (1961).
17. R. Zwanzig, in *Lectures in Theoretical Physics*, Vol. 3 (Wiley-Interscience, New York, 1961).
18. R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1975).
19. I. Prigogine, *Nonequilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1963).
20. R. Balescu, *Statistical Mechanics of Charged Particles* (Wiley-Interscience, New York, 1963).
21. H. H. Hasegawa and W. C. Saphir, *Phys. Rev. A* **46**:7401 (1992).
22. G. Nicolis, *Introduction to Nonlinear Science* (Cambridge University Press, Cambridge, 1995).
23. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North Holland, Amsterdam, 1981).

24. C. W. Gardiner, *Handbook of Stochastic Processes*, 2nd edition (Springer, Berlin, 1997).
25. R. Tabet, D. Saifaoui, A. Dezairi, and A. Raouak, *Eur. Phys. J. AP* **4**:329 (1998).
26. S. Benkadda, S. Kassibrakis, R. B. White, and G. M. Zaslavsky, *Phys. Rev. E* **55**:4909 (1997).
27. R. B. White, S. Benkadda, S. Kassibrakis, and G. M. Zaslavsky, *Chaos* **8**:757 (1998).
28. I. Dana and S. Fishman, *Physica D* **17**:63 (1985).
29. I. Dana, *Physica D* **39**:205 (1989).
30. C. F. F. Karney, A. B. Rechester, and R. B. White, *Physica D* **4**:425 (1982).
31. C. F. F. Karney, *Physica D* **8**:360 (1983).
32. J. D. Meiss and E. Ott, *Phys. Rev. Lett.* **55**:2741 (1985); *Physica D* **20**:387 (1986).
33. Y. H. Ichikawa, T. Kamimura, and T. Hatori, *Physica D* **29**:247 (1987).
34. T. Horita, H. Hata, R. Ishizaki, and H. Mori, *Prog. Theor. Phys.* **83**:1065 (1990).
35. R. Ishizaki, T. Horita, T. Kobayashi, and H. Mori, *Prog. Theor. Phys.* **85**:1013 (1991).
36. J. Klafter, G. Zumofen, and M. F. Shlesinger, *Fractals* **1**:389 (1993).
37. G. Zumofen and J. Klafter, *Europhys. Letters* **25**:565 (1994).
38. H. D. J. Abarbanel, *Physica D* **4**:89 (1981).
39. H. H. Hasegawa and D. J. Driebe, *Phys. Rev. E* **50**:1781 (1994).
40. I. Prigogine, C. George, F. Henin, and L. Rosenfeld, *Chemica Scripta* **4**:5 (1973).
41. B. Misra, I. Prigogine, and M. Courbage, *Physica A* **98**:1 (1979).
42. T. Petrosky and I. Prigogine, *Chaos, Solitons Fractals* **7**:441 (1995).
43. H. H. Hasegawa and W. C. Saphir, *Phys. Lett. A* **161**:471, 476 (1992).
44. H. H. Hasegawa and W. C. Saphir, *Phys. Lett. A* **171**:317 (1992).
45. D. J. Driebe, *Fully Chaotic Maps and Broken Time Symmetry* (Kluwer, Dordrecht, 1999).
46. D. Ruelle, *Phys. Rev. Lett.* **56**:405 (1986).
47. P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press, 1998).
48. N. G. van Kampen, *Physica* **21**:949 (1955); **23**:641 (1957).
49. K. M. Case, *Ann. Phys.* **7**:349 (1959).
50. T. Y. Petrosky, in: *Statistical Physics and Chaos in Fusion Plasmas*, C. W. Horton, Jr. and L. E. Reichl, eds. (Wiley-Interscience, New York, 1984).
51. P. V. Coveney and A. K. Evans, *J. Stat. Phys.* **77**:229 (1994).
52. A. K. Evans and P. V. Coveney, *Proc. Roy. Soc. London* **448**:293 (1995).
53. A. K. Evans and P. V. Coveney, *J. Phys. A: Math. Gen.* **31**:5887 (1998).